Instability of Yang-Mills equations and condensation of gluon field

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It is shown that in the presence of colored sources (such as heavy quarks and antiquarks) the gluon field becomes unstable and gluon condensation sets in, analogous to pion condensation in an electric field. The possible influence of this phenomenon on quark containment and hadron structure is discussed.

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We assume for simplicity $SU(2)$ symmetry and that there is no quark field—the quarks are regarded as external sources having only the color degree of freedom.

The initial Lagrangian is thus

$$L = \frac{1}{4g_0^2}(G^a_{\mu\nu})^2 + \frac{g_0}{2}\sum_k \delta(r - r_k)\chi^*_k \tau^a \chi_k A^a_0, \tag{1}$$

where τ^a are Pauli matrices acting on the color of quarks located at the points r_k:

$$G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \epsilon_{abc} A^b_\nu A^c_\mu .$$

Let immobile sources produce a sufficiently strong average field A^α_0 having a third color component $A_0 = A_0\delta_{\alpha 3}$. Since color enters in the nonlinear terms of the Yang-Mills equations in the form of vector products of colored vectors, the field A_0 does not enter in the equations for A^3_n and to explain the instability it is natural to assume $A^3_n = 0$. We introduce in lieu of A^α_n the charged components $A^\pm_n$ and A^0_n,

$$A^\pm_n = \frac{A^1_n \pm i A^2_n}{\sqrt{2}}, \qquad A^0_n = A^3_n = 0.$$

Simple algebraic manipulation then transform the Yang-Mills equation into:

$$\Delta A^\pm_n + (\omega \mp A_0)^2 A^\pm_n \mp \{A^\pm_m (A^+_n A^-_m - A^-_n A^+_m)\}_\omega = 0,$$

$$\Delta A_0 = \rho_0 + \Sigma\, 2(\omega - A_0) A^+_n A^-_n , \tag{2}$$

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where $\rho_0=\Sigma\chi^*_k\tau^3\chi_k\delta(r-r_k)$ and ω is the frequency of the field $A^{\pm}_n$.

Equations (2) have a simple physical meaning. The second term in the right-hand side of the second equation of (2) yields the change of the vacuum charge density in the presence of the field A_0. The first equation coincides with the equation for a charged vector boson in an electric potential A_0. To investigate the stability of this equation we can use the methods developed in[1] where the instability and the condensation of the field of scalar charged bosons in an electric field was investigated.

Instability of gluon field. We shall verify that if the field A_0 is large enough an instability sets in, i.e., an exponential growth of the field $A^{\pm}_n$. To this end we discard the nonlinear terms, multiply the second equation of (2) for A^+_n by $(A^+_n)^*$ and integrate over space. We denote by

$$\bar{B}=\frac{\int (A^+)^* B A^+ dV}{\int (A^+)^* A^+ dV}$$

the weighted mean value of the operator B. We obtain

$$\omega^2-2\omega\bar{A}_0+\overline{A_0^2}-\overline{p^2}=0, \tag{3}$$

where $\overline{p^2}$ is the mean value of the square of the momentum of the field A^+_n. It follows from (3) that

$$\omega=\bar{A}_0+\left[\bar{A}_0^2-\overline{A_0^2}+\overline{p^2}\right]^{1/2}.$$

When the condition $\overline{A_0^2}-\bar{A}_0^2+\overline{p^2}>0$ is satisfied, the field $A^{\pm}_n$ becomes unstable. We shall show that this condition is satisfied in the case of a quark—antiquark system at a sufficiently large distance R between the quarks. Indeed, in this case, as can be readily seen, $\bar{A}_0=0$; the quark containment means that the potential A_0 increases without limit with increasing R, whereas $\bar{p}^2\sim 1/R^2$.

Pion condensation. Since the instability gives rise to weakly inhomogeneous fields, the problem reduces to obtaining effective macroscopic equations for the classical field in a medium that is described by Yang-Mills equations.

Two types of linear effects determine the character of the equations of interest to us. First are effects that can be obtained by perturbation theory in the nonlinear terms of the "microscopic" Yang-Mills equations. These terms have a local character, i.e., they depend on the field intensities and not on the potentials, and can be taken into account by introducing into the Yang-Mills equations a dielectric constant $\epsilon(G_{\mu\nu})$ that depends on $(G^a_{\mu\nu})^2$. An analogous quantity for electrodynamics was introduced in[2] and its form, accurate to second order in the effective interaction constant $g^2(G_{\mu\nu})=g_0^2/\epsilon(G_{\mu\nu})$, agrees with the well known expression of Weisskopf, Heisenberg, and Euler. In the case of the Yang-Mills equations, an expression for $\epsilon(G_{\mu\nu})$ was obtained in[3]. We shall use below not the detailed course of $\epsilon(G_{\mu\nu})$, but only the fact that $\epsilon(G_{\mu\nu})$ decreases at large distances from the charge and consequently produces anti-screening. The term $(1/g_0^2)G_{\mu\nu}^2$ in the Lagrangian (1) is then replaced by $\epsilon G_{\mu\nu}^2$, as a result of which the nonlinear terms of Eqs. (2) are multiplied by ϵ, while ΔA^a is replaced by $\mathrm{div}\,\epsilon\nabla A^2$.

A much more substantial source of nonlinearity is the gluon-field condensation considered above, which is determined by the potentials A_0 and $A_n^{\pm}$.

In a quantum field-theoretical treatment of pion condensation in an electric field the Lagrangian considered differed from that corresponding to Eq. (2) only in the absence of the sign of n of the charged field and in the bringing out of the anharmonic terms. Following this paper, we neglect the influence of all the remaining degrees of freedom, other than the one corresponding to the condensate state. As shown in[1], the condensate stabilizes all the remaining degrees of freedom, and the change of their zero-point oscillations in the field A_0 exerts no substantial effect on the condensation conditions.

The ground state corresponds to a condensate with frequency $\omega=0$. The color structure and the coordinate dependence of the condensate field $A_n^a(r)$ are determined from the equations in (2), in which we introduce $\epsilon(G_{\mu\nu})$. Substitution of the matrix A_n^a in general form in (2) yields in the quasiclassical approximation [i.e., when $(\Delta+A_0^2)A_n^a$ is replaced by MA_n^a, where M is a numerical function]

$$A_3^a = 0, \quad A_n^a = \psi(r)[q_1\delta_{na} + q_2\epsilon_{na}],$$

where ϵ_{na} is a unit antisymmetrical tensor. With quasiclassical accuracy we have $\psi^2(r)(q_1^2+q_2^2)=A_0^2(r)$.

In a quantum treatment it is necessary, in accord with[1], to replace q_1 and q_2 by operators. For the condensate degrees of freedom we obtain the following Hamiltonian:

$$H = \frac{p_1^2 + p_2^2 + \tilde{\omega}^2(q_1^2 + q_2^2)}{2} + \frac{\lambda_1}{4}(q_1^2 + q_2^2) + \bar{A}_0 Z .$$

The problem has been reduced to a two-dimensional anharmonic oscillator. The condensate charge Z is an integer and is expressed in terms of the third projection of the oscillator moment in color space:

$$Z = q_1 p_2 - q_2 p_1 .$$

Here $\tilde{\omega}^2=\bar{A}_0^2-\bar{A}_0^2+\overline{p^2}<0$, $\lambda_1=\frac{1}{2}\int\psi^4 dV$, and ψ is defined by the equation

$$\Delta\psi(r) + \nabla\ln\epsilon\,\nabla\psi + A_0^2\,\psi_0 - \frac{(q_m q_m q_n)_{01}}{(q_n)_{01}}\psi^3 = 0. \qquad (6)$$

Let $\psi^3(r)$ be normalized to unity, and then $(q_1^2+q_2^2)$ has a large mean value that increases with R, since $\lambda_1 \sim 1/V$, therefore

$$(q_m q_m q_n)_{01} \cong \langle q_m q_m \rangle (q_n)_{01} \equiv \xi^2 (q_n)_{01}.$$

As a result, Eq. (6) does not depend on n or on the choice of the states 0 and 1. We can write down an equation also for the eigenstates of the continuous spectrum, and these should go over outside the range of action of the field A_0 into the zero-point oscillations of the free gluon field. It is the condensate field which ensures the positiveness of ω^2 (i.e., the stability). Accurate to small corrections (that do not contain R), the condensate energy is (see[11])

$$E = \overline{A}_0 Z + \frac{Z^2}{2\xi^2} + \frac{\tilde{\omega}^2 \xi^2}{2} + \frac{\lambda_1 \xi^4}{4}.$$

Minimizing the energy with respect to ξ and Z, we get

$$\xi^2 = \frac{\overline{A_0^2} - \overline{p^2}}{\lambda_1}, \qquad Z = -\overline{A}_0 \frac{\overline{A_0^2} - \overline{p^2}}{\lambda_1}, \qquad E = -\frac{(\overline{A_0^2} - \overline{p^2})^2}{4\lambda_1} \tag{7}$$

At $\overline{A}_0=0$ the energy minimum corresponds to $Z=0$. Thus, the mean-squared condensate field increases with R like A_0^2, whereas the average field $A_n^a = \langle q_m \rangle \psi = 0$ is strictly equal to zero. We present also an expression for the density of the gluon charge

$$\rho = -A_0 \langle A_n^a A_n^a \rangle = -2A_0 \xi^2 \psi^2(r). \tag{8}$$

Thus, the equation for A_0 takes the form

$$\nabla \ln \epsilon \nabla A_0 + \Delta A_0 = \rho_0/\epsilon - 2A_0 \xi^2 \psi^2(r). \tag{9}$$

The solution of Eqs. (6) and (9) together with the equations for the zero-point oscillations in a condensate field makes it possible, we hope, to obtain theoretically the properties of the hadron "bag" and, in particular to establish whether it is a string.

The instability of the gluon field is directly connected with the instanton oscillations of the vacuum. Indeed, consider a gluon-field fluctuation corresponding to formation of positive and negative charges. At short distances between the charges, besides the initial gluon field, only the field A_0 is produced. However when the fluctuations become large enough (when the charges move far enough apart) an instability sets in and an additional gluon field is produced. When the fluctuations die out the system can go over into a state with $G_{\mu\nu}=0$, but the fields A_0 and A_n^a that differ from zero, and this in fact corresponds to instanton fluctuations.

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