Topological singularities on the surface of an ordered system

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The methods of relative homotopic groups are used to classify topologically stable defects on the boundary of an ordered system. Surface defects in nematics and in superfluid He^3 are considered.

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Topology methods are extensively used of late to investigate various spatially inhomogeneous structures in ordered systems. Homotopic groups were used for the classification of singularities in an order-parameter field.[1,2] Individual homotopic groups were used to classify nonsingular topologically stable solitons and textures in vessels of various shapes.[3] It was assumed in[3] that no surface singularities are present on the vessel boundary. Such singularities, however, can play an important role in the dynamics of a system, for example the motion of "boojums" over the boundary of a vessel with He^3-A leads to continuous damping of the persisting current in the ring.[4,5] An attempt was made in[6] to classify the surface defects by using the homotopic group $\pi_1(\tilde{R})$ for the singular points on the surface and the group $\pi_0(\tilde{R})$ for the singular lines on the surface, where $\tilde{R}$ is the manifold of internal states of the system on its surface. This classification, however, is not quite complete, and it will be shown below that the singular points on the surface are described by the relative homotopic group $\pi_2(R,\tilde{R})$ (R is the manifold of internal states of the volume of the system). In some cases, for example in a nematic, this leads to additional singular points that are not described by $\pi_1(\tilde{R})$.

The topological analysis carried out in[6] is the following: A singular point on the surface is surrounded by a contour that is mapped in the $\tilde{R}$ space—the region of the variation of the order parameter on the surface of the system. $\tilde{R}$ is a subspace of the region of the variation of the order parameter in the volume R, because the boundary condition narrow down the region of variation of the order parameter on the boundary. It is stated in[6] that the group $\pi_1(\tilde{R})$–the classes of the mappings of the contour in $\tilde{R}$–describe singular points on the surface. However, among the elements of the group $\pi_1(\tilde{R})$ there can be contained elements of the group $\pi_1(R)$. Let a be such an element. Then the contour surrounding the singular point of class a cannot be continuously contracted into a point not only on the surface of the system, but also in the volume. Consequently, this singular point is the end of a singular line passing through the volume. If we consider only isolated singular points, then it is necessary to take into account only those elements of $\pi_1(\tilde{R})$ which correspond to zero in $\pi_0(R)$. This means that we must find the kernel of the homomorphism $\pi_1(\tilde{R})\rightarrow\pi_1(R)$.

This homomorphism takes place if the contour surrounding the singular point is shifted from the surface into the interior of the system.

In addition to the singular points, which are described by the group

$$\operatorname{kern}\,(\pi_1(\tilde{R}) \to \pi_1(R)) \tag{1}$$

there can exist on the surface some of the singular points that have arrived from the volume but have remained topologically stable on the surface.

To classify the singular points of both types we surrond the investigated singular point by a hemisphere lying in the volume, and let the boundary of the hemisphere lie on the surface of the systems. The classes of the mappings of this hemisphere in R and of its boundaries in $\tilde{R}$ form a relative homotopic group $\pi_2(R,\tilde{R})$. The elements of this group give all the classes of the singular points on the system surface. As is known from topology, there exists an exact sequence of homomorphisms that connect this group with the known groups:

$$\pi_2(\tilde{R}) \to \pi_2(R) \to \pi_2(R, \tilde{R}) \to \pi_1(\tilde{R}) \to \pi_1(R). \tag{2}$$

From the definition of the exact sequence of the homomorphism (the image of any of the homomorphisms of the sequence is the kernel of the next homomorphism) it follows that the group $\pi_2(R,\tilde{R})$ contains a normal divisor that is isomorphic to the factor-group

$$\pi_2(R)/\operatorname{im}(\pi_2(\tilde{R}) \to \pi_2(R)). \tag{3}$$

Here im $(A \to B)$ denotes the image of the homomorphism $A \to B$.

The factor group of the $\pi_2(R,\tilde{R})$ group is isomorphic to the group (1) with respect to this normal divisor. In many cases the group $\pi_2(R,\tilde{R})$ is merely the direct product of groups (1) and (3). In this case the system contains the following: the singular points that are described by the group (1); those of the singular points that have arrived from the volume and remained topologically stable (they are described by groups), as well as combinations of both types (they are described by products of elements of the two groups).

By way of example, we consider a nematic. The space of the order parameter in the volume is:

$$R = S^2/Z_2, \quad \pi_2(R) = Z, \quad \pi_1(R) = Z_2.$$

The boundary conditions require that the director vector lie in the plane of the surface, therefore

$$\tilde{R} = S^1/Z_2, \quad \pi_1(\tilde{R}) = Z, \quad \pi_2(\tilde{R}) = 0.$$

By considering the homomorphisms that take place when the space $\tilde{R}$ is expanded to R, we get

$$\operatorname{kern}(\pi_1(\tilde{R}) \to \pi_1(R)) = Z.$$

These are disclinations on the surface with even Frank indices; it is known that in the volume they can be deformed into a nonsingular configuration. Next,

$$\pi_2(R)\,/\,\mathrm{im}(\pi_2(\tilde{R}) \to \pi_2(R)) = \pi_2(R) = Z.$$

That is to say, all the singular points are topologically stable in the volume and on the boundary. Thus, the relative homotopic group that describes all the surface singular point in the nematic, is

$$\pi_2(R, \tilde{R}) = Z + Z. \tag{4}$$

In the case of the A phase of He^3 with dipole interaction, where

$$R = SO_3,\quad \tilde{R} = S^1 \times Z_2,\quad \pi_2(R) = 0,\ \pi_1(R) = Z_2,\ \pi_2(\tilde{R}) = 0,\ \pi_1(\tilde{R}) = Z$$

we obtain the known result

$$\pi_2(R, \tilde{R}) = \mathrm{kern}(\pi_1(\tilde{R}) \to \pi_1(R)) = Z. \tag{5}$$

The singular points on the surface–"boojums"–are vortices with even numbers of superfluid-velocity circulation quanta.

We can analogously investigate the singular lines on the surface of a system. The corresponding relative group $\pi_1(R,\tilde{R})$ for them, is determined by the elements of the group

$$\mathrm{kern}(\pi_0(\tilde{R}) \to \pi_0(R))$$

and the elements of the factor group

$$\pi_1(R)\,/\,\mathrm{im}(\pi_1(\tilde{R}) \to \pi_1(R)).$$

In the case of a nematic, where $\pi_0(R) = \pi_0(\tilde{R}) = 0$, we have

$$\pi_1(R, \tilde{R}) = \pi_1(R)\,/\,\pi_1(R) = 0.$$

Consequently, it has no stable singular lines on the surface. In the case of He^3-A with dipole interaction, where $\pi_0(R) = 0$ and $\pi_0(\tilde{R}) = Z_2$, we obtain likewise the known result[6]

$$\pi_1(R, \tilde{R}) = \pi_0(\tilde{R}) = Z_2.$$

These are the so-called boundaries of islands with overturned l.[4,5]

The next task is the investigation of the singular point and lines on the boundary with account taken of the external and internal fields that alter the region where the order parameter varies and consequently alter the classification of the topologically stable defects and establish various levels of their stability, as was done in[3] for solitons.

In conclusion, I wish to thank P.W. Anderson for sending the preprint[6] that stimulated the writing of this paper.

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