Canonical equations of the hydrodynamics of rotating superfluid ^{4}He

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Nondissipative hydrodynamics of rotating superfluid ^{4}He is considered. Canonical variables are introduced, and the quantities describing the systems are expressed in their terms. The Hamiltonian technique is used to deduce the canonical equations of motion. A complete system of nonlinear hydrodynamic equations is derived and the conservation laws are obtained in explicit form.

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It is known that in rotating superfluid ^{4}He the curl of the superfluid velocity is not equal to zero, owing to the presence of the vortices. This makes it necessary to generalize the hydrodynamics of superfluid ^{4}He. Nonlinear hydrodynamics for this case was considered on the basis of the conservation laws by Bekarevich and Khalatnikov.[1] The equations, however, contained an indeterminacy due to the presence of a reactive term in the equation for the superfluid velocity. This indeterminacy was eliminated by comparison with the results of Hall and Vinen.[2] In the present paper, on the basis of the Hamiltonian formalism, we obtain the canonical equations of rotating superfluid ^{4}He.

Both superfluid and normal motions exist in superfluid ^{4}He. Connected with them are respectively the superfluid momentum density $\mathbf{j}$ and the relative normal momentum density $\mathbf{p}$; the latter is connected with the excitation and vanishes in the limit as $T\to 0$. The total momentum density is

$$\mathbf{g} = \mathbf{j} + \mathbf{p}. \tag{1}$$

By a Galilean transformation from the coordinate system in which $\mathbf{v}_s = 0$, we obtain for the energy density

$$E = \frac{\rho v_s^2}{2} + (\mathbf{g} - \rho \mathbf{v}_s)\mathbf{v}_s + \epsilon(\mathbf{p}, \rho, s, \vec{\omega}), \tag{2}$$

where ρ is the mass density, s is the entropy density, and $\vec{\omega}$ is the local angular velocity connected with the vortices. The differential of the energy density is given by

$$dE = \mathbf{g}\,d\mathbf{v}_s + \mathbf{v}_s d(\mathbf{j} - \rho\mathbf{v}_s) + (\mu + \frac{v_s^2}{2})d\rho + T\,ds + \vec{\lambda}\,d\vec{\omega}, \tag{3}$$

where $\mathbf{v}$ is the normal velocity, T is the temperature, and μ is the chemical potential. The pressure is

$$P = (\mathbf{v} - \mathbf{v}_s, \mathbf{p}) + \mu\rho + \vec{\omega}\vec{\lambda} + sT - \epsilon \tag{4}$$

The canonical equations of the hydrodynamics of superfluid ^{4}He were derived by Pokrovskii and Khalatnikov.[3,4] To write out the Hamiltonian equations it is necessary to know the structure of the Hamiltonian

$$\mathcal{H} = \int d^3 r H. \tag{5}$$

The energy density E becomes equal to the density of the Hamiltonian H if all the quantities on which it depends are expressed in terms of the canonical variables. In particular, for non-rotating superfluid ^{4}He we have[3,4]

$$\mathbf{j} = -\rho\nabla\alpha \quad , \quad \mathbf{p} = -s\nabla\beta - f\nabla\gamma. \tag{6}$$

Here (ρ,α), (s,β) and (f,γ) are pairs of canonically conjugate variables. The last pair consists of the Clebsch variables, which are needed for the description of the independent component of p.

It is seen from (6) that $-\vec{\nabla}\alpha$ plays the role of the superfluid velocity $\mathbf{v}^s$. In rotating ^{4}He, where $\vec{\nabla}\times\mathbf{v}^s\neq 0$, this expression must be modified. We start with the analogy with superconductors, where the presence of the vortices is connected with the presence of a magnetic field. Accordingly, we introduce for rotating ^{4}He a "vector potential" **a** such that

$$\mathbf{v}^s = -\vec{\nabla}\alpha + \mathbf{a}. \tag{7}$$

The analog of the magnetic field of superconductors is

$$\vec{\omega} = [\vec{\nabla}\times a] \equiv [\vec{\nabla}\times\mathbf{v}^s]. \tag{8}$$

It is necessary also to introduce a variable **d**, which is canonically conjugate to **a** and is the analog of the electric displacement vector. Connected with the vortices is a momentum density determined by a "Poynting vector." It is natural to include it in the superfluid momentum density, since, in contrast to the excitation momentum, it does not vanish in the limit as $T\to 0$. Ultimately[1)]

$$\mathbf{j} = \rho\mathbf{v}^s - [\mathbf{d}\times\vec{\omega}]. \tag{9}$$

Now all the quantities on which E depends are expressed in terms of the canonical variables, and we can write down the canonical equations:

$$\frac{\partial \rho}{\partial t} = - \frac{\delta \mathcal{H}}{\delta a} \equiv -\vec{\nabla} \mathrm{g} , \tag{10}$$

$$\frac{\partial \mathbf{d}}{\partial t} = - \frac{\delta \mathcal{H}}{\delta \mathbf{a}} \equiv - \mathrm{g} - \left[\vec{\nabla} \left(\left[\mathbf{d} \times \mathrm{v}_s \right] + \vec{\lambda} \right) \right], \tag{11}$$

$$\frac{\partial s}{\partial t} = - \frac{\delta \mathcal{H}}{\delta \beta} \equiv - \vec{\nabla} (s \, \mathrm{v}), \tag{12}$$

$$\frac{\partial t}{\partial t} = - \frac{\delta \mathcal{H}}{\delta \gamma} \equiv - \vec{\nabla} (f \, \mathrm{v}), \tag{13}$$

$$\frac{\partial a}{\partial t} = \frac{\delta \mathcal{H}}{\delta \rho} \equiv \mu + \frac{\mathrm{v}_s^2}{2} \tag{14}$$

$$\frac{\partial \mathbf{a}}{\partial t} = \frac{\delta \mathcal{H}}{\delta \mathbf{d}} \equiv \left[\mathrm{v}_s \times \vec{\omega} \right] , \tag{15}$$

$$\frac{\partial \beta}{\partial t} = \frac{\delta \mathcal{H}}{\delta s} \equiv T - \mathrm{v} \vec{\nabla} \beta, \tag{16}$$

$$\frac{\partial \gamma}{\partial t} = \frac{\delta \mathcal{H}}{\delta t} \equiv - \mathrm{v} \vec{\nabla} \gamma . \tag{17}$$

From (14) follows the transport equation

$$\frac{\partial \omega}{\partial t} = \left[\vec{\nabla} \left[\mathrm{v}_s , \vec{\omega} \right] \right] . \tag{18}$$

It is necessary to add to the system (10–17) also the analog of Maxwell's equation

$$\vec{\nabla} \mathbf{d} = \rho, \tag{19}$$

which is a first integral, as seen from (10) and (11). Using (19), we get from (10–17)

$$\frac{\partial \mathbf{p}}{\partial t} + \vec{\nabla}_i \left(v_i \, \mathbf{p} \right) = - p_i \vec{\nabla} v_i - s \vec{\nabla} T \tag{20}$$

$$\frac{\partial j}{\partial t} + \nabla_i (v_{si} (\mathrm{j} - \rho \mathrm{v}_s) + g_i \mathrm{v}_s) = p_i \vec{\nabla} v_{si} - \rho \vec{\nabla}\mu + [[\vec{\nabla}\times\vec{\lambda}]\times\vec{\omega}]. \tag{21}$$

Equations (10), (12), (18), (20), and (21) constitute the complete system of equations of the hydrodynamics of rotating superfluid ^{4}He. This system leads to the energy conservation law

$$\frac{\partial E}{\partial t} + \vec{\nabla} Q = 0, \tag{22}$$

where the energy flux density is

$$\mathrm{Q} = \mathrm{v}_s(\mathrm{j} - \rho \mathrm{v}_s, \mathrm{v}_s) + \left(\mu + \frac{\mathrm{v}_s^2}{2}\right)\mathrm{g} + \left[\left[\omega\times\frac{\mathrm{g}}{\rho}\right]\times\vec{\lambda}\right] + (\mathrm{p}\,\mathrm{v})\,\mathrm{v} + Ts\,\mathrm{v}, \tag{23}$$

We can also formulate the momentum conservation law

$$\frac{\partial g_i}{\partial t} + \nabla_k \Pi_{ik} = 0, \tag{24}$$

where the stress tensor is

$$\Pi_{ik} = P\delta_{ik} + p_i v_k + v_{si} g_k - \lambda_i \omega_k + v_{sk}(j_i - \rho v_{si}) \tag{25}$$

It is symmetrical by virtue of the invariance of E to rotations.

We note that in contrast to[1] we now have independent equations for $\vec{\omega}$ an j. Accordingly, the number of kinetic terms in the equations is increased: in particular, in contrast to[1], the relative velocity and $\vec{\nabla}\times\vec{\lambda}$ enter in the kinetic terms independently.

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[1)]This expression is gauge-invariant, and we therefore use it instead of the standard $-\rho\vec{\nabla}\alpha - d_i\vec{\nabla}Q_i$, which differs from (9), when account is taken of (19), by the total divergence.

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