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Topological susceptibility in Yang-Mills theory in the vacuum correlator method

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We calculate the topological susceptibility of the Yang-Mills vacuum using the field correlator method. Our estimate for the $SU(3)$ gauge group, $\chi^{1/4} = 196(7)\,\mathrm{MeV}$, is in a very good agreement with the results of recent numerical simulations of the Yang-Mills theory on the lattice.

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The topology of the QCD vacuum and the hadron phenomenology are intimately related. The celebrated example is given by the Witten-Veneziano formula [1] which provides a solution to the "$U(1)$ problem" by utilizing the explicit breaking of the $U(1)$ axial symmetry in the large-N QCD. In a simplest case of N_f massless quarks the formula relates the mass of the η' meson,

$$m^2_{\eta'} = \frac{2N_f}{F^2_\pi}\,\chi\,, \tag{1}$$

to the non-perturbative susceptibility

$$\chi = \lim_{V\to\infty}\frac{1}{V}\left\langle (Q-\langle Q\rangle)^2\right\rangle = \int \mathrm{d}^4 z\,\langle q(0)q(z)\rangle, \tag{2}$$

of the topological charge

$$Q = \int \mathrm{d}^4 x\, q(x)\,, \tag{3}$$

calculated in the pure $SU(N)$ Yang-Mills theory, and to the pion decay constant F_π (note that $\langle Q\rangle = 0$ due to CP–invariance of the Yang-Mills vacuum). An exact value of the susceptibility χ is the subject of intensive numerical studies in the lattice gauge theories [2, 3].

The local density $q(x)$ of the topological charge is

$$q(x) = \frac{g^2}{32\pi^2}\epsilon_{\alpha\beta\mu\nu}\mathrm{Tr}\,(F_{\alpha\beta}(x)F_{\mu\nu}(x)), \tag{4}$$

where $F_{\mu\nu} = F^a_{\mu\nu}T^a$ is the non-Abelian field strength tensor with the components

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu\,, \tag{5}$$

and T^a, $a = 1,\dots,N^2-1$ are the generators of the $SU(N)$ gauge group normalized by the standard relation, $\mathrm{Tr}\,T^aT^b = \delta^{ab}/2$, and $f^{abc} = -2i\mathrm{Tr}\,[T^a,T^b]T^c$ is the totally asymmetric structure constant of the group.

In this paper we estimate the topological susceptibility (2) in the formalism of gauge-invariant non-local correlators [4] which is reviewed in Ref. [5, 6]. We work in Euclidean space-time suitable for comparison of our results with the ones obtained in numerical simulations of the lattice Yang-Mills theory. A basic element of the field correlator method is a non-local quantity,

$$G_{\mu\nu}(x;x_0) = \Phi(x_0;x)F_{\mu\nu}(x)\Phi(x;x_0), \tag{6}$$

where $\Phi(x;x_0) \equiv \Phi^\dagger(x_0,x)$ is an ordered exponent of the integral over the gauge field $A_\mu = T^a A^a_\mu$ (the Schwinger line). The integration is taken along the oriented path $\mathcal{C}_{x,x_0}$ which stretches between the points x_0 and x of the space-time,

$$\Phi(x;y) = \mathcal{P}\exp\Big[-ig\int_x^y A_\mu(z)\,dz_\mu\Big]\,, \tag{7}$$

where $\mathcal{P}$ denotes path ordering. Under the $SU(N)$ gauge transformation Ω the quantity (7) changes as

$$\Phi(x;y) \to \Omega(x)\Phi(x;y)\Omega^\dagger(y)\,.$$

Due to this property the non-local object (6) transforms essentially locally,

$$G_{\mu\nu}(x,x_0) \to \Omega(x_0)G_{\mu\nu}(x,x_0)\Omega^\dagger(x_0)\,,$$

making it possible to construct various (non-local) gauge invariant objects of the general form

$$D^{(n)}_{\mu_1\dots\nu_n} \propto \mathrm{Tr}\,[G_{\mu_1\nu_1}(x_1;x_0)\dots G_{\mu_n\nu_n}(x_n;x_0)]. \tag{8}$$

One may think about this quantity as of the n-point correlation function of the field strength tensors (5) covariantly shifted from the "reference point" x_0 to the points x_i, $i = 1,\dots,n$ along appropriate paths. The simplest nontrivial quantity of the kind (8) is the two-point correlator (hereafter we omit the superscript $n = 2$)

$$D_{\mu\nu\alpha\beta}(x_1, x_2; x_0) = \frac{g^2}{N} \mathrm{Tr}\,[G_{\mu\nu}(x_1, x_0)\, G_{\alpha\beta}(x_2; x_0)]. \quad (9)$$

One may choose the paths connecting the points x_i, $i = 0, 1, 2$ as straight lines and put the point x_0 exactly on a straight line between the points x_1 and x_2. Then the correlator (9) becomes a function of a single variable $z = x_1 - x_2$.

At zero temperature the correlator (9) can be parameterized by two scalar functions D and D_1

$$D_{\mu\nu\alpha\beta}(z) = \Bigl(\delta_{\mu\alpha}\delta_{\nu\beta} - \delta_{\mu\beta}\delta_{\nu\alpha}\Bigr)\, D(z^2) + \\ + \frac{1}{2}\Bigl[\frac{\partial}{\partial z_\mu}(z_\alpha \delta_{\nu\beta} - z_\beta \delta_{\nu\alpha}) - (\mu \leftrightarrow \nu)\Bigr]\, D_1(z^2). \quad (10)$$

Perturbatively, both functions are divergent as $1/|z|^4$ at short distances, $|z| \to 0$. However, besides the perturbative part the structure functions possess also a non-perturbative part which is relevant to the long–distance physics. The lattice simulations [7, 8] indicate that the structure functions can be well described as follows

$$\begin{aligned} D(z^2) &= A_0\, e^{-|z|/T_g} + \frac{b_0}{|z|^4}\, e^{-|z|/\lambda}, \\ D_1(z^2) &= A_1\, e^{-|z|/T_g} + \frac{b_1}{|z|^4}\, e^{-|z|/\lambda}, \end{aligned} \quad (11)$$

where the first terms in both expressions correspond to a non-perturbative contribution while the last two terms contain the perturbative parts. The functions D and D_1 are parameterized by the correlation lengths T_g and λ, and by the prefactors A_i and b_i, $i = 0, 1$.

On general grounds one may expect that the correlators (8) should depend not only on the points x_i, but also on the auxiliary variables such as x_0 and shapes of the paths $\mathcal{C}_{x_n, x_0}$ entering the phase factors (7) used in Eq. (6). Perturbative corrections coming from the Schwinger lines may provide an additional contribution to the correlation lengths. However, a physical non-perturbative observable – if expressed in terms of the correlators (8) only-should be independent of both the shape of the Schwinger lines and the ultraviolet cutoff.

In what follows we consider the quadratic correlator (9) for which various numerical estimations are available [7 – 9]. The numerical results were obtained in lattice simulations with the use of the so-called cooling method which makes it possible to get rid of short-range lattice artifacts while leaving the long-range physics intact. The cooling removes ultraviolet suppression of long Schwinger lines $\Phi(x, y)$, which may originate from perturbative fluctuations. As a result of the (soft) cooling, the correlation lengths T_g and λ in the correlators (11) do not depend on the ultraviolet cutoff both in pure Yang-Mills theory [7] and in QCD with dynamical quarks [7]. Similar results were obtained in Ref. [9] with the use of the so called renormalization group smoothing which is an alternative to the cooling.

Another subtlety of the lattice results is the dependence of the correlators on the shape of the Schwinger line connecting the points x_0, x_1 and x_2 in the definition of the correlator (9). It turns out that the dependence of the prefactors A_i and b_i, $i = 0, 1$ is rather strong, while the correlation lengths T_g and λ are insensitive with respect to the variations of the shape of the line [10]. Thus our results should be understood with the prescription that the perturbative ultraviolet corrections are subtracted and the Schwinger line itself is chosen to take a most natural shape forming straight path stretched between the points x_1 and x_2 in the definition of the correlation function (9). In this prescription the correlator is clearly independent of the reference point x_0 provided it is located on the Schwinger line stretched between the points x_1 and x_2.

It is argued [5] that in the Yang-Mills theory and in QCD the dominant contribution to various observables is given by the lowest bilocal correlator. This observation is often called as the "Gaussian dominance". The validity of the Gaussian approximation is supported, for example, by the argument, that the contribution of the bilocal correlations to the interactions between colored particles dominates the contribution coming from the higher-order correlators. The dominance could be interpreted either in stochastic picture (any $n > 2$ correlator contributes much less then the binomial one) or in the coherent picture which postulates that there is a strong cancelation between all $n > 2$ correlator so that the result is given only by the binomial contribution [11]. We take an advantage of the stochastic scenario assuming that all higher order correlators are suppressed with respect to the leading Gaussian contribution.

In the Gaussian vacuum all n-point correlators of field strengths are assumed to be factorized into the bilocal correlators. In particular, this means that all odd correlators are zero in the Gaussian approximation. The factorization of even correlation functions goes according the scheme [5]:

$$\langle G^{a_1}(1) G^{a_2}(2) \ldots G^{a_{2n}}(2n)\rangle \sim (\delta^{a_1 a_2} \ldots \delta^{a_{2n-1} a_{2n}}) \times \\ \times \langle \mathrm{Tr}\, G(1) G(2)\rangle \ldots \langle \mathrm{Tr}\, G(2n-1) G(2n)\rangle + \\ + \text{all permutations}, \quad (12)$$

where the color arguments of the field strength tensors are written in a short form.

The quadratic correlation function (2) of the topological densities involves the correlator of the four field strengths operators located in two points of the space time. A related four-point correlator of field strengths can be calculated in the Gaussian approximation according to the factorization scheme described above:

$$\begin{aligned}&\langle F^a_{\alpha\beta}(0)F^b_{\gamma\delta}(0)F^c_{\mu\nu}(z)F^d_{\rho\theta}(z)\rangle = \\ &= \langle F^a_{\alpha\beta}(0)F^b_{\gamma\delta}(0)\rangle \langle F^c_{\mu\nu}(z)F^d_{\rho\theta}(z)\rangle + \\ &+ \langle F^a_{\alpha\beta}(0)F^c_{\mu\nu}(z)\rangle \langle F^b_{\gamma\delta}(0)F^d_{\rho\theta}(z)\rangle + \\ &+ \langle F^a_{\alpha\beta}(0)F^d_{\rho\theta}(z)\rangle \langle F^b_{\gamma\delta}(0)F^c_{\mu\nu}(z)\rangle \equiv \\ &\equiv \frac{4}{g^4}\left(\frac{\mathrm{N}}{\mathrm{N}^2-1}\right)^2 \Big[\delta^{ab}\delta^{cd}\, D_{\alpha\beta\gamma\delta}(0) D_{\mu\nu\rho\theta}(0) + \\ &+ \delta^{ac}\delta^{bd}\, D_{\alpha\beta\mu\nu}(z)\, D_{\gamma\delta\rho\theta}(z) + \\ &+ \delta^{ad}\delta^{bc}\, D_{\alpha\beta\rho\theta}(z)\, D_{\gamma\delta\mu\nu}(z)\Big]. \end{aligned} \tag{13}$$

In this calculation we identified the field strength tensors $F_{\mu\nu}$ – used in the definition of the topological charge density (4) – with the covariantly transformed field strength tensors $G_{\mu\nu}$, Eq. (6), used in the definition of the gauge–invariant bilocal correlators (9). We used the observation that the four-point correlation function (13) involves only two distinct points $x_1 = 0$ and $x_2 = z$. As a result, the expression (13) can be factorized into two bilocal correlation functions with the same Schwinger lines. In the axial gauge, $z_\mu A_\mu(z) = 0$, the Schwinger line (7) is equal to unity, and therefore one arrives to the identification $G \equiv F$, which is valid only in this gauge. However, since both the topological density (4) and the bilocal correlator (6) are the gauge-invariant quantities, our results – obtained with in the axial gauge – must be gauge-independent.

The correlation function of two topological densities (4) is calculated with the help of Eqs. (10) and (13):

$$\begin{aligned}\langle q(0)q(z)\rangle &= \frac{3}{32\pi^4}\frac{N^2}{N^2-1}\Big[D(z)+D_1(z)\Big] \times \\ &\times \Big\{2\Big[D(z)+D_1(z)\Big] + z_\mu \frac{\partial D_1(z)}{\partial z_\mu}\Big\}. \end{aligned} \tag{14}$$

In a physical language the use of the factorization prescription (13) in the pseudoscalar channel implies that we have associated (a tower of) pseudoscalar glueballs – which mediate the interaction between the pseudoscalar q-probes – by two non-perturbatively dressed gluons. For a more appropriate treatment of the glueballs in the field correlator formalism see Ref. [12].

It is important to notice that the two-point correlation function (14) in Euclidean space-time must always be negative for non-zero z,

$$\langle q(0)q(z)\rangle < 0 \quad \text{for} \quad |z| > 0. \tag{15}$$

due to a reflection positivity property and a pseudoscalar nature of the topological charge [13]. Using the parametrization for the bilocal correlators (11) as well as the fitting results obtained with the help of the lattice simulations [7, 8, 14] one can immediately check that the negativity requirement is indeed satisfied for large enough distances (typically, for $|z| \gtrsim 1.5\,\mathrm{fm}$). However, at smaller distances the correlator (14) becomes positive due to large positive contributions coming from the perturbative parts of the functions $D(z)$ and $D_1(z)$. These parts are given in Eq. (11) by the terms proportional to the factors b_0 and b_1.

It is important to stress that the positivity of the correlator (14) (calculated with the parameters taken from the lattice data [7, 8, 14]) has an apparent inconsistency with the asymptotic freedom of the Yang-Mills theory. In fact, at small distances the physics must be dominated by the perturbative corrections. For example, in the tree order (here $\alpha_s = g^2/(4\pi)$) one has

$$D^{\mathrm{tree}}(z) = 0, \qquad D_1^{\mathrm{tree}}(z) = \frac{2\alpha_s}{\pi}\frac{N^2-1}{N}\frac{1}{|z|^4}, \tag{16}$$

which leads to the known negative-valued result,

$$\langle q(0)q(z)\rangle^{\mathrm{tree}} = -\frac{3\alpha_s^2}{4\pi^6}\frac{N^2-1}{|z|^8} < 0. \tag{17}$$

One can check that perturbative corrections to $D(z)$ and $D_1(z)$ structure functions [15] do not spoil this result. Note that we do not discuss here non-perturbative physics at short distances since the corresponding terms must anyway be small compared to the leading perturbative result.

Moreover, in order for the correlator (15) to be negative at small distances, the parameters b_0 and b_1 of the parametrization (11) must satisfy the condition

$$b_0 < b_1. \tag{18}$$

However, neither of available results of the lattice simulations [7, 8, 14] of the pure Yang-Mills theory and QCD agrees with the requirement (18).

The observed inconsistency with the perturbative part of the lattice results and the reflection positivity does not however undermine non-perturbative calculations utilizing the field correlator method. A plausible explanation of the contradiction may be related to the cooling procedure used in the lattice simulations. In fact, the cooling may affect the short range part of the bilocal correlators, leading to a modification of the coefficients b_0 and b_1 as compared with their values in the uncooled vacuum. In Ref. [7] it is clearly demonstrated that the field correlator at a fixed short separation, e.g. $|z| \approx 0.3\,\mathrm{fm}$, in the course of the cooling increases by a factor of 3 from its initial negative value to a positive value at a plateau. Moreover, the fits of the numerical data for the correlators were done at higher separations z, and therefore we do not expect that the fitting results represent correctly the short distance physics. As a consequence, there is no contradiction between the lattice results [7, 8, 14] and the reflection positivity (15).

Since the aim of this paper is to evaluate the non-perturbative contribution to the susceptibility of the topological charge, our results should not be affected by uncertainties in determination of the perturbative part of the bilocal field correlators.

Another important remark is that in our calculations given below we neglect the contact term [13] in the correlator of the local topological densities (14). This term should appear as a singular δ–like function at zero separations, $z = 0$, in order to reconcile the apparent positivity of the susceptibility (2) with the negativity requirement (15). Unfortunately, the contact term cannot be accounted in our calculations because we are constrained (by results of the lattice simulations) to use the particular prescription (11) of the field correlators (10). Moreover, we expect that in the quoted lattice calculations this term cannot be accessible anyway because these numerical calculations use the cooling method which must destroy (or affect substantially) the short distance physics. On the other hand, we know that the (soft) cooling does not affect the chiral and bulk topological properties of the QCD vacuum [16]. In terms of the correlation function (14) the latter observation may indicate that in the course of the cooling the non-perturbative part of the zero-distance singularity is not destroyed literally but it is rather shifted towards longer distances. If true, this means that we can neglect the $z = 0$ singularity provided we use only the non-perturbative $z \neq 0$ part which is obtained with the use of the cooling method. With this justification in mind we now continue with explicit calculations.

We use the data of Ref. [14] where the long-range tail – left intact by the cooling procedure – was fitted only by the non-perturbative terms of Eqs. (11). The fits (with vanishing perturbative part, $b_0 = b_1 = 0$) provide us with the following results [14]

$$\begin{aligned} D^{\mathrm{NP}}(0) &\equiv A_0 = 3.62(19)\,\Lambda_L^4 = 0.212(11)\,\mathrm{GeV}^4\,, \\ D_1^{\mathrm{NP}}(0) &\equiv A_1 = 1.23(7)\,\Lambda_L^4 = 0.072(4)\,\mathrm{GeV}^4\,, \\ T_g &= \frac{1}{183(3)\,\Lambda_L} = \frac{1}{0.900(14)\,\mathrm{GeV}} = 0.222(4)\,\mathrm{fm}\,, \end{aligned} \tag{19}$$

written in various units for the sake of convenience. Here the superscript "NP" stands for "non-perturbative", and $\Lambda_L = 4.92\,\mathrm{MeV}$ denotes a lattice renormalization scale chosen in Ref. [14].

Substituting Eq. (14) into Eq. (2) and taking non-perturbative parts of the correlators (11) we arrive to

$$\chi = \frac{9}{64\pi^2}\frac{N^2}{N^2-1} D^{\mathrm{NP}}(0)\Big[D^{\mathrm{NP}}(0) + D_1^{\mathrm{NP}}(0)\Big] T_g^4. \tag{20}$$

Finally, a direct evaluation of the susceptibility (20) with the help of the non-perturbative part of the field correlators gives us:

$$\chi^{1/4}_{\mathrm{theor}} = 196(7)\,\mathrm{MeV}. \tag{21}$$

Here we used the parameters (19) from Ref. [14] as a numerical input.

It is remarkable that our result (21) – based on the direct evaluation of the topological susceptibility – coincides (within the small error bars) with the numerical value

$$\chi^{1/4}_{\mathrm{lattice}} = 193(9)\,\mathrm{MeV}. \tag{22}$$

obtained recently in lattice simulations of pure $SU(3)$ gauge theory [3]. One should note, however, that the error in our estimation (21) refers to numerical errors of the lattice data (19) and does not reflect any systematic uncertainty related our assumptions we implied in the course of derivation of Eq. (20).

In conclusion, we stress that in our calculation we have used the particular prescriptions related to (i) the renormalization of the ultraviolet divergences (self-energy) associated with the Schwinger lines, (ii) the path-dependence of the field correlation functions, and (iii) the presence of the divergent contact term in the correlators of the topological charge densities. After a proper treatment of these subtle issues we arrive to the analytical result (20), which for $SU(3)$ gauge theory gives the numerical value (21) in a good agreement with the lattice result (22).

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