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Modulation theory of quantum tunneling into a Calogero-Sutherland fluid

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Quantum hydrodynamics of interacting electrons with a parabolic single particle spectrum is studied using the Calogero-Sutherland model. The effective action and modulation equations, describing evolution of periodic excitations in the fluid, are derived. Applications to the problem of a single electron tunneling into the FQHE edge state are discussed.

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It is known since 60's [1] that Landau Fermi liquid theory is not applicable to one dimensional metals. The thermodynamic of one dimensional electronic system is explained by Tomonaga-Luttinger (TL) model [2, 3]. This model is a simplified version of a quantum hydrodynamics that takes into account only a single mode of sound excitation (phonons) of an electronic fluid. This approximation is equivalent to replacing a true electronic single-particle spectrum by the linear spectrum of Dirac's fermions. Though the TL model accounts for thermodynamics it fails for effects that involve breaking of the particle-hole symmetry. In other words only the diagonal part of kinetic coefficient matrix [4] can be found using this approach, while all non-diagonal terms vanish in this approximation.

Among phenomena missed by the TL approximation are thermopower, photovoltaic effect and Coulomb drag with a small momentum transfer. To solve any of this problems one needs to use quantum hydrodynamic without making one mode approximation.

The problem of single particle spectrum curvature and electron-electron interaction was recently addressed in Ref. [5], who proposed to mimics a nonlinear spectrum by two types of electrons with different Dirac's spectra. This approach agrees with perturbative expansion of exact results for Calogero-Sutherland (CS) model [6].

Here, we would follow a different route of "exact" bosonization, originally developed in Ref. [7] for the sake of matrix models. As expected [8], a curvature of fermionic spectrum leads to the cubic in density terms in a bosonic Hamiltonian. Though formally a bosonic description is achieved, the resulting theory is non linear. This renders this method extremely difficult for applications and no new results based on it are reported, as yet.

To gain an understanding of one dimensional hydrodynamics we study the special case of electrons with singular $1/r^2$ interaction. This problem is known as the Calogero-Sutherland model (CS) [9] and has numerous applications in condensed matter and nuclear physics [10]. Being exactly solvable [11, 12] CS model posses an infinite number of integrals of motion. That render its hydrodynamics special and easier to analyze.

The connection between the hydrodynamic excitations of the CS fluid and correlation functions of various operators is not straightforward. The simplest known example is the soliton, a solution of hydrodynamical equation of motion propagating without changing its shape. In original description it corresponds to the many body excitation, called anyon. Anyons obey a special "fractional" statistics, which is neither Fermi nor Bose like. The correlation function of anyons is similar to the one of free fermions, with the values of Fermi velocity and momentum renormalized by interaction [13].

In order to find other correlation function (such as electrons' Green function) one needs to solve the system of saddle point equations, that are the continuity and the Euler equation for CS fluid. This is a very difficult problem. It is considerably simplified if the solution is a periodic wave with slowly changing wave parameters. In this case, the original hydrodynamic equations may be replaced by Whitham's modulation equations [14]. To put it simple, the TL model describes particle-hole excitations of fermionic problem as sound modes (vibrations) of one dimensional harmonic string. The modulation theory describes particle-hole excitations as vibration of an *anharmonic* string.

The modulation technique is well developed for various models of dissipationless hydrodynamics. In particular for the Benjamin-Ono (BO) equation [15, 16], that describes dynamics of internal waves in stratified fluids of great depth. As it was stressed recently [17], this two problems are mathematically related. Remarkably, the classical BO equation appears in FQHE [18], governing the evolution of a semiclassical wave packets containing a large number of fermions. It is therefore not surprising that some formal aspects of the derivations of modulation theory for the BO and the CS models are similar. The final results of these two models are nevertheless different.

The structure of this work is a following: starting with the microscopic description of the CS model we pass into its hydrodynamics (the technical details are given in the Appendix). Next, we consider periodic density excitation of the CS model and derive its evolution equations. Finally, we apply this machinery to the problem of a single electron tunneling into the CS liquid.

The microscopic Hamiltonian of CS model is given by

$$H=-\frac{\hbar^2}{2m}\sum_{i=1}^{2N}\partial_i^2+\left(\frac{\pi}{L}\right)^2\sum_{i>j}^{2N}\frac{\lambda(\lambda-1)}{\sin^2\left(\frac{\pi}{L}(x_i-x_j)\right)}. \quad (1)$$

Here λ is a strength of particle interaction, m is an electron mass, x_i is a coordinate of the particle on a circle with the perimeter L. Now on we use the convention $\hbar=1, m=1$.

On large scales and long time CS model is described by hydrodynamical Hamiltonian

$$H=\int dx\left[\frac{1}{2}\rho v^2+U[\rho]\right] \quad (2)$$

(the details of derivation that follows Ref. [7, 19] are presented in Appendix). The whole specific of CS model is incorporated in potential energy term

$$U=\frac{\pi^2\lambda^2}{6}\rho^3-\frac{\pi\lambda(\lambda-1)}{2}\rho^H\rho_x+\frac{(\lambda-1)^2}{8}\frac{(\rho_x)^2}{\rho}, \quad (3)$$

where the Hilbert transform is defined as

$$\rho^H(x)=\frac{1}{\pi}P\int dx'\frac{\rho(x')}{x'-x}. \quad (4)$$

The conventional TL model can be obtained by expansion of the Hamiltonian (2) up to second order in fluctuations (due to conservation laws the velocity v and density operators $\delta\rho$ are of the same order) and neglecting the high gradient terms.

It is convenient to reformulate the problem in Lagrangian description, defining the action

$$S[\rho,v]=\int dxdt\left[-v\partial_x^{-1}\partial_t\rho-\frac{1}{2}\rho v^2-U[\rho]\right]. \quad (5)$$

This action reaches its minimum provided saddle point equations are satisfied: the continuity

$$\rho_t+\partial_x(\rho v)=0 \quad (6)$$

and Euler equation

$$v_t+vv_x+w_x=0. \quad (7)$$

Here the enthalpy

$$w=\left(\frac{\delta U}{\delta\rho}\right)=\frac{\pi^2}{2}\rho^2+\pi\rho_x^H-\frac{1}{4}\partial_x\left(\frac{\rho_x}{\rho}\right)-\frac{1}{8}\left(\frac{\rho_x}{\rho}\right)^2. \quad (8)$$

The dependence on the interaction constant in the limit of strong interaction ($\lambda\gg 1$) had been eliminated by the rescaling ($\tilde{x}=x/\lambda, \tilde{\rho}=\rho\lambda, \tilde{t}=t/\lambda$). Now on we use the rescaled coordinates, omitting the tilde.

To derive modulation equation it is convenient to integrate the velocity field out

$$L=\int dx\left(\frac{(\partial_x^{-1}\partial_t\rho)^2}{2\rho}-\frac{\pi^2}{6}\rho^3+\frac{\pi}{2}\rho^H\rho_x-\frac{1}{8}\frac{\rho_x^2}{\rho}\right). \quad (9)$$

This Lagrangian has one-periodic density wave solutions [20]

$$\rho_s(\theta)=\rho_0+\frac{k}{2\pi}\frac{\sinh(a)}{\cosh(a)-\cos(\theta)}, \quad (10)$$

where $\theta=kx-\omega t$. The dispersion relation between amplitude of the wave

$$A=\frac{k}{2\pi}\frac{1}{\sinh(a)} \quad (11)$$

and a wave vector is determined by

$$\tanh(a)=\frac{\pi\rho_0k^3}{\omega^2-\pi^2\rho_0^2k^2-\frac{k^4}{4}}. \quad (12)$$

To proceed further we replace the Polychronakos solution with a modulated one

$$\theta(x,t)=k(x,t)x-\omega(x,t)t \quad (13)$$

and allow ρ_0 to depend on coordinate and time as well. Strictly speaking this is no longer the minimum of the action (5). However under the condition the modulation technique works, this solution minimizes the action "in average". We define Lagrangian averaged over one period of oscillations

$$\bar{L}=\int_0^{2\pi}\frac{d\theta}{2\pi}L[\rho_s(\theta)]. \quad (14)$$

After some straightforward, though lengthy calculation, one finds a Lagrangian of anharmonic string

$$2\bar{L}=\frac{\omega^2}{2\pi k}-\omega\rho_0+\frac{\gamma^2}{\rho_0}\left(1-\frac{k^2}{2\omega}\right)+\gamma k-\frac{k^3}{24\pi}-\frac{\pi^2\rho_0^3}{3}. \quad (15)$$

Here $\gamma \equiv -\partial_x^{-1}\partial_t\rho_0$. Applying the least action condition to the averaged Lagrangian

$$\left(\frac{\partial\bar{L}}{\partial\omega}\right)_t=\left(\frac{\partial\bar{L}}{\partial k}\right)_x,$$
$$\left(\frac{\partial\bar{L}}{\partial\gamma}\right)_t=\left(\frac{\partial\bar{L}}{\partial\rho_0}\right)_x, \quad (16)$$

one obtains modulation eqs. for CS model

$$\left(\frac{\omega}{\pi k}+\frac{\gamma^2}{2\rho_0}\frac{k^2}{\omega^2}\right)_t+\left(\frac{\gamma^2}{\rho_0}\frac{k}{\omega}+\frac{\omega^2}{2\pi k^2}+\frac{k^2}{8\pi}\right)_x=0,$$
$$\left(\frac{\gamma}{\rho_0}\left[1-\frac{k^2}{2\omega}\right]\right)_t+\left(\frac{\gamma^2}{2\rho_0^2}\left[1-\frac{k^2}{2\omega}\right]+\frac{\pi^2\rho_0^2}{2}\right)_x=0,$$
$$\partial_t\rho_0+\partial_x\gamma=0,\quad \partial_t k+\partial_x\omega=0. \quad (17)$$

These eqs. govern the dynamics of the wave parameters in the density excitations propagating through CS fluid.

Next, we apply this theory to study an evolution of a distortion caused by adding one electron to the fluid. This problem is a prototype for a quantum *tunneling* from the normal metal into the edge state of FQHE.

An added electron causes a density fluctuation that splits into two chiral parts, moving in the opposite directions. Using Riemann invariants [21]

$$u=v+\pi\rho, \quad (18)$$
$$\bar{u}=v-\pi\rho \quad (19)$$

one approximates eqs.(6), (7) by

$$u_t+uu_x=0,\quad \bar{u}_t+\bar{u}\bar{u}_x=0. \quad (20)$$

The tunneling at point $x=0$ corresponds to the initial conditions $u_0(\xi)=\epsilon/(\xi^2+\epsilon^2)$, (where ϵ has a scale of an electron wave length). To study right chiral sector we pass into a reference frame moving with a sound velocity to the right ($\xi=x-\pi\bar{\rho}t$). Solving Hopf eq.(20) by Godograph method we find an implicit solution

$$u=u_0(\xi-ut). \quad (21)$$

Solving cubic equation(21), we find an explicit solution

$$u=\begin{cases}\frac{\xi}{t},\ 0<\xi<\xi_- \\ \\ \frac{\epsilon}{\xi^2},\ \xi>\xi_+\end{cases},$$

where $\xi_-\equiv 3(\epsilon t/4)^{1/3}, \xi_+\equiv t/\epsilon$ are tailing and leading edge coordinates. Inside the interval ($\xi_-<\xi<\xi_+$) solution of eq. (21) is multivalued and we denote three different branches by $f_1>f_2>f_3$. The multi-validity of solution reflects a wave breaking phenomena. The single value solution inside the interval ($\xi_-<\xi<\xi_+$) is restored by keeping the second-order spatial derivatives in eq. (8). An elegant way of dealing with second-order gradient terms is to use the modulation technique developed above. Let us assume that for ($\xi_-<\xi<\xi_+$) density is an oscillating function satisfying modulation eqs.(17) with a proper boundary condition [22].

At the boundaries, the particle density found from Hopf equation, should match the density, averaged over the period of fluctuations, inside the oscillating interval

$$\langle\rho\rangle=\rho_0+\frac{k}{2\pi}. \quad (22)$$

In addition, the amplitude of the oscillation vanished at the trailing edge

$$A=0,\quad \xi=\xi_- \quad (23)$$

and the wave vector vanishes at the leading edge

$$k=0,\quad \xi=\xi_+\,. \quad (24)$$

It is convenient to define phase velocity $c=\omega/k$ and hydrodynamic velocity $V=\gamma/\rho_0$. In a new variables eq.(23) can be rewritten as

$$c=s+\frac{k}{2}. \quad (25)$$

Solving eqs.(17), we find that electron tunneling into CS liquid excites density wave with phase velocity and wave vector given by

$$c=\frac{f_1}{2}\simeq\frac{\xi}{2t},$$
$$k=f_1-f_2\simeq\frac{2\epsilon}{t}\sqrt{\frac{t}{\epsilon\xi}-1},$$
$$\rho_0=\bar{\rho}+\frac{f_3}{2\pi}\simeq\bar{\rho}+\frac{\epsilon^2}{2\pi\xi}. \quad (26)$$

As we see, the excitation that follow electron tunneling are quite different from the anyon tunneling [13]. The later results in the creation of a single soliton propagating with a constant velocity through the fluid. Electron's tunneling causes a spreading density evolution that consist of many picks with linearly increasing phase velocity.

The large number of oscillations experienced by the fluid density and the smooth dependence of wave parameters a posteriori justifies the validity of modulation technique. One can check, using eq. (23), that solution eq. (26) respects the particle conservation and accounts for the half electrons moving to the right (solution for another half of electron moving to the left differs by sign change of a sound velocity).

The periodic density wave that develops after the tunneling can be viewed as a superposition of anyons, with individual anyone corresponding to the picks of the density oscillations. Therefore eq.(26) describe a *decay* of an electron into a large number of quasiparticles of CS model. This process may be detected by measurement the time dependence of a current that follows the tunneling event. Unlike the fractional charge measurements in shot noise experiments [23, 24], the tunneling discussed above is from normal metal to the FQHE state. Therefore the fractionalization of the elementary charge does not show up in a low frequency shot noise, but in the finite frequency noise of a current pulse.

In this work we developed modulation theory for the CS model. We applied this theory to the problem of a single-electron tunneling. We found an evolution of a current pulse that followed the tunneling event.

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Appendix:

Derivation of Hydrodynamical Theory. Consider a ring geometry. Electron's coordinate is represented by a complex variable $z_n = Le^{i\theta_n}$ where θ is an angle along the circle of radius L. In this variables the Hamiltonian eq.(1) is given by

$$H = \frac{1}{2}\sum_{j=1}^{2N}(z_j\partial_j)^2 + \sum_{i\neq j}^{2N}\frac{\lambda(\lambda-1)}{|z_i-z_j|^2}. \tag{27}$$

The ground state of the Hamiltonian wave function of (27) can be found exactly

$$\Psi_0 = \left(\prod_{i=1}^{2N} z_i\right)^{-\lambda(2N-1)/2} |\Delta|^{\lambda-1}\Delta, \tag{28}$$

where

$$\Delta = \prod_{i<j}^{2N}(z_i - z_j) \tag{29}$$

is Vandermonde determinant. Excited states are given by

$$\Psi_\kappa = \Psi_0 J_\kappa, \tag{30}$$

where Jack polynomials J_κ are parameterized by partition κ. The problem had been reduced to the properties of the new bosonic Hamiltonian

$$H_{\rm B} = \Psi_0^{-1} H \Psi_0 \tag{31}$$

that acts in the Hilbert space of symmetric wave functions. It is given by

$$H_{\rm B} = \sum_{i=1}^{2N} D_i^2 + \lambda \sum_{i<j}^{2N} \frac{z_i+z_j}{z_i-z_j}(D_i - D_j), \tag{32}$$

where $D_i = z_i\partial_i$. Aiming for a second quantization one defines so called collective variables

$$p(\theta) = \sum_{i=1}^{2N}\delta(\theta-\theta_i),\ \ p_k = \int_0^{2\pi} d\theta e^{ik\theta}p(\theta),$$
$$p(z) = \sum_{k=-\infty}^{\infty} z^{k-1}p_{-k}\,. \tag{33}$$

In terms of collective variables the bosonic Hamiltonian can be rewritten as [19]

$$H_{\rm B} = \frac{1}{2}\sum_{m,n=-N}^{N} mnp_{n+m}\frac{\partial^2}{\partial p_n\partial p_m} + (1-\lambda)\sum_{n=-N}^{N} n^2 p_n\frac{\partial}{\partial p_n} + \frac{\lambda}{2}\sum_{m=0}^{N-1}\sum_{n=1}^{N-m}(n+m)\left[p_np_m\frac{\partial}{\partial p_{n+m}} + p_{-n}p_{-m}\frac{\partial}{\partial p_{-n-m}}\right]. \tag{34}$$

So far the transformation have been exact. Passing to the hydrodynamic limit ($N\to\infty$, $\to\infty$ $2N/L\to\bar{\rho}$ one arrives to eq.(2); x is a coordinate along the circle ($x = \frac{L}{2\pi}\theta$), the linear density $\rho(x) = \frac{2\pi}{L}\rho(\theta)$. The modes of velocity operator are defined as

$$v_n = 2\pi\left(-n\frac{\partial}{\partial p_{-n}} + \frac{1}{2}p_n\mathrm{sgn}(n)\right). \tag{35}$$

It is easy to see that these definitions are consistent with a standard commutation [25]

$$[v(x), \rho(y)] = -i\delta'(x-y). \tag{36}$$

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