ANALYTICITY OF THE FORWARD ELASTIC SCATTERING AMPLITUDE WITH RESPECT TO THE SQUARE OF THE SCATTERED-PARTICLE MASS

B. V. Geshkenbein and B. L. Ioffe
Institute of Theoretical and Experimental Physics
Submitted 12 October 1973
ZhETF Pis. Red. 18, No. 10, 646 — 649 (20 November 1973)

It is proved on the basis of the results of [1] that the function $T(\nu\sqrt{q'^2/q^2}, q'^2)$ (where $T(\nu, q^2)$ is the forward scattering amplitude) is analytic in $q'^2$ in the complex $q'^2$ plane with a cut $(0, \infty)$. It is shown that knowledge of the functions $w_i(\nu, q^2)$ at $q^2 < 0$, which describe the electroproduction process, makes it possible to reconstruct the forward scattering amplitude of a virtual photon with $q^2 > 0$.

The consequences ensuing from the causality condition were investigated in [1] for the problem of the forward scattering of a particle with squared mass $q^2$ by a spinless target (or by a fermion in the case when the amplitude for scattering without spin flip is averaged over the projections of the fermion spin). It was proved that the causality condition makes some of the functions introduced in [1] analytic in $q^2$. The results can be used to analyze the electroproduction process on a nucleon (as was indeed done in [1]), or any other problem where the analyticity properties of the amplitude with respect to the squared mass must be known. We show in the present article the connection between the functions introduced in [1] and the forward-scattering amplitudes, and prove by the same token that the forward scattering amplitudes are analytic in the mass squared.

We consider for simplicity the scattering of a neutral spinless particle with momentum q by another spinless particle with momentum p. The imaginary part of the forward scattering amplitude $T(\nu, q^2)$ can be expressed in the form

$$\frac{1}{\pi} T(\nu, q^2) \equiv w(\nu, q^2) - \rho(q^2) = \frac{1}{2\pi}\sum_n \langle p|J(0)|n\rangle \langle n|J(0)|p\rangle \times (2\pi)^4\delta^4(p+q-p_n) - \frac{1}{2\pi}\sum_n \langle 0|J(0)|n\rangle\langle n|J(0)|0\rangle (2\pi)^4\delta^4 \times (q-p_n) |\langle p|p\rangle|^2, \quad (1)$$

where $J(x)$ is the current of the scattered particle and $\nu = pq \geq 0$. At $q^2 < 0$, $\rho(q^2) \equiv 0$ and $w(\nu, q^2) = \pi^{-1}\mathrm{Im}(\nu, q^2)$. (The function $w(\nu, q^2)$ at $q^2 < 0$ is analogous to the corresponding functions involved in the electroproduction problem.) As shown in [1], to satisfy the causality principle, i.e., for the matrix element of the current commutator $\langle p|[J(x), J(0)p\rangle$ to vanish at $x^2 < 0$, it is necessary and sufficient that the function

$$F\left(\frac{\nu^2}{q^2}, q'^2\right) = 2\int_{-q'^2/2}^{\infty} \frac{\nu' q'^2 - 2[\nu^2(q'^2/q^2) - \nu'^2](\sqrt{\nu'^2 - q'^2} - \nu')}{\nu'^2 - \nu^2(q'^2/q^2)} w(\nu', q'^2)\, d\nu' \quad (2)$$

be analytic in the variable $q'^2$ in a complex plane with a cut from 0 to $\infty$. It is assumed that the integral in (2) converges (generalizations to include the case when this assumption is not valid are considered in [1]). The discontinuity on the cut of $F(\nu^2/q^2, q'^2)$ at $q'^2 = q^2 > 0$ is equal to $F(\nu^2/q^2, q^2 + i\varepsilon) - F(\nu^2/q^2 - i\varepsilon) = 2\pi i w(\nu, q^2)$. We write down (2), introducing a subtraction at the point $\nu^2/q^2 = m^2$ (m is the target mass)

$$F\left(\frac{\nu^2}{q^2}, q'^2\right) - F(m^2, q'^2) = (q'^2)^2 2\left(\frac{\nu^2}{q^2} - m^2\right)\int_{-q'^2/2}^{\infty} \nu' d\nu' \times$$

$$\times \frac{w(\nu', q'^2)}{\left(\nu'^2 - \nu^2 \frac{q'^2}{q^2}\right)(\nu'^2 - m^2 q'^2)} . \qquad (3)$$

Let us compare (3) with the dispersion relation in $\nu$ for the forward scattering amplitude $T(\nu, q'^2)$ at $q'^2 < 0$, also written down with one subtraction at the point[1] $\nu = m\sqrt{q'^2}$

$$T(\nu, q'^2) - T(m\sqrt{q'^2}, q'^2) = 2(\nu^2 - m^2 q'^2) \int_{-q'^2/2}^{\infty} \nu' d\nu' \frac{w(\nu', q'^2)}{(\nu'^2 - \nu^2)(\nu'^2 - m^2 q'^2)} . \qquad (4)$$

It follows from (3) and (4) that

$$F\left(\frac{\nu^2}{q^2}, q'^2\right) - F(m^2, q'^2) = q'^2 \left[ T\left(\nu\sqrt{\frac{q'^2}{q^2}}, q'^2\right) - T\left(m\sqrt{q'^2}, q'^2\right) \right] \qquad (5)$$

Relation (5) is proved for $q'^2 < 0$. However, since $F(\nu^2/q^2, q'^2)$ is an analytic function of $q'^2$ in the entire complex plane with the cut $(0, \infty)$, it follows that (5) also holds true in the entire region of analyticity of $F(\nu^2/q^2, q'^2)$, i.e., the difference $T(\nu\sqrt{q'^2/q^2}, q'^2) - T(m\sqrt{q'^2}, q'^2)$ is an analytic function of $q'^2$ in this region. By starting from the definition of $T(\nu, q'^2)$, we can readily show that $T(m\sqrt{q'^2}, q'^2)$ is an analytic function of $q'^2$ in the same region. In fact, in the laboratory frame,

$$T(m\sqrt{q'^2}, q'^2) = i \int d^4x \, e^{i\sqrt{q'^2} x_0} \theta(x_0) < p | [J(x), J(0)] | p > \qquad (6)$$

(we have discarded the inessential terms with equal-time commutators, and the analyticity of $T(m\sqrt{q'^2}, q'^2)$ in $q'^2$ in the complex $q'^2$ plane with the cut $(0, \infty)$ is evident from (6)). We have proved by the same token that the scattering amplitude $T(\nu\sqrt{q'^2/q^2}, q'^2)$ with real $\nu^2/q^2 > 0$ is an analytic function of $q'^2$ in the entire complex $q'^2$ plane with the cut $(0, \infty)$.

Let us explain the result in the language of Feynman diagrams. Let the problem be to find a function of the variables $\nu$ and $q'^2$ that is 1) analytic in the complex plane $q'^2$ with a cut $(0, \infty)$ and 2) its discontinuity at $q'^2 = q^2 > 0$ is equal to $w(\nu, q^2)$. The amplitude $T(\nu, q'^2)$ cannot be such a function, since it has singularities at $q'^2 > 0$, owing to the diagrams with singularities in the s channel $s = q'^2 + 2\nu + m^2$. However, if the argument $\nu$ in $T(\nu, q'^2)$ is replaced by $\nu\sqrt{q'^2/q^2}$, then (and only then) the denominator in the right-hand side of (4) does not vanish and $T(\nu\sqrt{(q'^2/q^2)}, q'^2)$ is real at $q'^2 < 0$. This accounts for the appearance of $\sqrt{q'^2}$ in $T(\nu\sqrt{q'^2/q^2}, q'^2)$. By examining the Feynman diagrams we can also show that the discontinuity on the cut in $q'^2$, due to $T(\nu\sqrt{q'^2/q^2})$ is equal to $w(\nu, q^2)$ at $q'^2 = q^2 > 0$ and $\nu > 0$. For $T(\nu, q^2)$, we use in the proof the integral representation usually called the Deser-Gilbert-Sudarshan representation (DGSR) [3 - 6], which, as is well known, holds in any order of perturbation theory. According to the DGSR, $T(\nu\sqrt{q'^2/q^2}, q'^2)$ can be written in the form

$$T\left(\nu\sqrt{\frac{q'^2}{q^2}}, q'^2\right) = - \int_0^1 d\alpha \int_0^\infty da \, h(\alpha, a) \times$$

$$\times \left[ \frac{1}{q'^2 + 2\alpha\nu\sqrt{\frac{q'^2}{q^2}} - a} + \frac{1}{q'^2 - 2\alpha\nu\sqrt{\frac{q'^2}{q^2}} - a} \right] . \qquad (7)$$

Taking the discontinuity of (7) at $q'^2 = q^2 \pm i\varepsilon$, we have

$$\frac{1}{2\pi i}\left[T\left(\nu\sqrt{\frac{q^2+i\epsilon}{q^2}},\ q^2+i\epsilon\right)-T\left(\nu\sqrt{\frac{q^2-i\epsilon}{q^2}},\ q^2-i\epsilon\right)\right]_{\epsilon\to 0} =$$

$$= \int_0^1 d\alpha \int_0^\infty da\, h(\alpha,\ a)[\epsilon(q^2+\alpha\nu)\,\delta(q^2+2\alpha\nu-a)+\epsilon(q^2-\alpha\nu)\,\delta(q^2-2\alpha\nu-a)]. \qquad (8)$$

On the other hand, we get from the DGSR the following expression for $w(\nu,\ q^2)$

$$w(\nu, q^2) = \int_0^1 d\alpha \int_0^\infty da\, h(\alpha,\ a)[\epsilon(q_0-p_0\alpha)\delta(q^2+2\alpha\nu-a)+\epsilon(q_0-p_0\alpha)\,\delta\times$$

$$\times(q^2-2\alpha\nu-a)].$$

It is easy to see that at $q^2 > 0$ and $\nu > 0$ formulas (8) and (9) are equal to each other, as well as to (1).

All the foregoing results can be directly applied to the scattering of a virtual photon by a nucleon, where at $q^2 < 0$ one measures the functions $w_1(\nu,\ q^2)$ and $w_2(\nu,\ q^2)$ in the process $e + N \to e +$ hadrons. The relations (2) - (5) hold in this case for the amplitudes $T_1(\nu,\ q^2)$ and $T_2(\nu,\ q^2) = [(\nu^2 - q^2)/q^2]T_1(\nu,\ q^2)$. In accord with the statements made above, it follows that from the known functions $w_1(\nu,\ q^2)$ and $w_2(\nu,\ q^2)$ at $q^2 < 0$ the amplitude $T_2(\nu,\ q^2)$ can be reconstructed uniquely at all real $q^2$ (since $T_2(m\sqrt{q^2},\ q^2) = 0$, which corresponds to the absence of subtractions in the dispersion relation in $\nu$ for $T_2(\nu,\ q^2)$, and the amplitude $T_1(\nu,\ q^2)$ accurate to a function of $q^2$, is analytic in the complex $q^2$ plane with a cut from 0 to $\infty$.[2])

---

[1])We consider the dispersion relation in $\nu$ at $q'^2 < 0$. As is well known [2], the proof of the dispersion relation when the square of the mass is negative entails no difficulty).

[2])In practice, in view of the instability in the reconstruction of an analytic function from its values on the curve outside the cuts, one can obtain with sufficient accuracy only certain integrals of $T(\nu,\ q^2)$ with respect to $q^2$, with weighting functions of $q^2$.


[1] B. V. Geshkenbein and A. I. Komech, Yad. Fiz. 18, 914 (1973) [Sov. J. Nuc. Phys. 18, No. 4 (1974)].
[2] N. N. Bogolyubov and D. V. Shirkov, Vvedenie v teoriyu kvantovannykh polei (Introduction to the Theory of Quantized Fields), Gostekhizdat, 1957 [Interscience, 1960].
[3] V. D. Skarzhinskii, Diploma thesis, Moscow State University, 1957.
[4] V. Ya. Fainberg, Zh. Eksp. Teor. Fiz. 36, 1503 (1959) [Sov. Phys.-JETP 9, 1066 (1959)].
[5] S. Deser, W. Gilbert, and E.C.S. Sudarshan, Phys. Rev. 117, 266 (1960).
[6] M. Ida, Progr. Theor. Phys. 23, 1151 (1960).
[7] N. Nakanishi, Phys. Rev. D4, 2571 (1972).


## WAVE FUNCTIONS OF THE PARTON MODEL


I. S. Shapiro
Institute of Theoretical and Experimental Physics


Submitted 12 October 1973
ZhETF Pis. Red. 18, No. 10, 650 — 654 (20 November 1973)



It is shown that the scaling of the cross sections of deeply inelastic processes means that the partons are localized in a three-dimensional Euclidean r-space that is the Fourier transform of the rapidities. The onset of scaling is determined by the form of the wave function of the hadron in this space.


1. In the momentum representation, the Schrödinger wave function of the hadron

$$\phi_n(k_1\epsilon_1;\ \ldots\ k_n\epsilon_n;\ M)$$

depends on the momenta $k_i$ and the energies $\varepsilon_i$ of the partons, with