Contribution of instantons to the correlation functions of a Heisenberg ferromagnet

I. V. Frolov and A. S. Shvarts

Moscow Engineering Physics Institute
(Submitted 23 June 1978)
Pis'ma Zh. Eksp. Teor. Fiz. **28**, No. 5, 274–276 (5 September 1978)

The quantum fluctuations of instantons are calculated in the Heisenberg model of a ferromagnet.

PACS numbers: 75.10.Jm

The theory of a Heisenberg ferromagnet (two-dimensional nonlinear σ model) is analogous in some respects to the Yang-Mills theory. This theory, in particular, postulates the existence of instantons, which were found in[1]; the quantum fluctuations of instantons with unity topological charge were calculated in Refs. 2 and 3. Some partial results on the quantum fluctuations of instantons with topological charge 2 were obtained in Ref. 3. In the present article we obtain the quantum fluctuations of instantons with arbitrary topological charge q. The calculation methods were developed in Refs. 4 and 5.

The theory in question is specified by the functional

$$S = \frac{1}{2f_0} \int (\partial_\mu \mathbf{n})^2 dx dy,$$

where $\mathbf{n} = (n_1, n_2, n_3)$ is the field, which is specified in two-dimensional Euclidean space and satisfies the condition $|\mathbf{n}| = 1$. (This functional has the meaning of the energy of a Heisenberg ferromagnet or of the Euclidean action of the σ model). We change over to the complex variables $z = x + iy$; $w = (n_1 + in_2)(1 + n_3)^{-1}$. In terms of these variables,

$$S = 2 f_0^{-1} \int (1 + |w|^2)^{-2} (|\partial_z w|^2 + |\partial_{\bar z} w|^2)\, i dz d\bar z\,; \qquad (1)$$

$$f_0 S - 4\pi q = 4 \int (1 + |w|^2)^{-2} |\partial_{\bar z} w|^2\, i dz d\bar z\,, \qquad (2)$$

where $\partial_z = \frac{1}{2}(\partial_x - i\partial_y)$; $\partial_{\bar z} = \frac{1}{2}(\partial_{\bar x} + i\partial_y)$. It is seen from (2) that the minimum of S on fields having a topological charge $q > 0$ is equal to $4\pi q f^{-1}$ and is reached on fields of the type

$$w = c \prod_{1 \leqslant i \leqslant q} (z - a_i) \prod_{1 \leqslant j \leqslant q} (z - b_j)^{-1} \qquad (3)$$

(on instantons). The correlation functions of a Heisenberg ferromagnet (Euclidean Green's functions of the σ model) are represented in the form

$$\int \Phi(\mathbf{n}) \exp(-S) \prod d\mathbf{n}(x) \Big/ \int \exp(-S) \prod d\mathbf{n}(x), \qquad (4)$$

 0021-3640/78/170249-03$00.60

where it can be assumed, for example, that $\Phi(\mathbf{n}) = n_{\alpha 1}(x_1)...n_{\alpha_k}(x_k)$. Our problem is to find the contribution of the instantons with topological charge q and calculate, by the saddle-point method, the functional integrals contained in (4). In this calculation, only terms quadratic in the deviations from instantons are retained in the functional S. The determination of the functional integral over the directions orthogonal to the manifold of the instantons reduces to an investigation of an infinite-dimensional determinant; this investigation is the main difficulty encountered in the study of the instanton contribution. We shall show that the contribution made to (4) by instantons with topological charge q is given by the formula

$$I_q = f^{-q}\nu^{2q}\exp(-4\pi q f^{-1})\int \Phi(a,b,c)d\mu,$$

where f is the physical coupling constant, ν is the normalization point,

$$d\mu = \prod_{j<k}|a_j - a_k|^2 \prod_{l<m}|b_l - b_m|^2 \prod_{r,s}|a_r - b_s|^{-2}$$

$$\times (1+|c|^2)^{-2}\prod_p (i da_p d\bar{a}_p)\prod_t (i db_t a\bar{b}_t)\, i\, dc d\bar{c}$$

$\Phi(a,b,c) = \Phi(a_1,...,a_q,b_1,...,b_q,c)$ is obtained by substituting the instanton (3) in $\Phi(\mathbf{n})$, and integration is with respect to the parameters of the instantons (the constant factor in the expression for I_q will be omitted). It is most convenient to calculate I_q by considering fields on a sphere of radius R and letting R then go to infinity. We represent first I_q in the form

$$I_q = f^{-2q}\nu^{2q}\exp(-4\pi q f^{-1})\int \Phi(a,b,c)(\det{}'\Delta)^{-1/2}d\mu_0 .$$

Here $d\mu_0$ is a measure generated on the instanton manifold by the natural Riemannian metric on this manifold:

$$d\mu_0 = \det(M_{pt})\prod_{j<k}|a_j - a_k|^2\prod_{l<m}|b_l - b_m|^2\prod_{r,s}|a_r - b_s|^2$$

$$\times |c|^{4q}\prod_j (i da_j d\bar{a}_j)\prod_k (i db_k d\bar{b}_k) i dc d\bar{c}, \qquad (5)$$

$$M_{pt} = \int \rho^{-2}\bar{z}^p z^t\sqrt{g}\, i dz d\bar{z} \quad (0 \leqslant p,\ t \leqslant 2q),$$

$$\rho = (1+|w|^2)\prod_j |z - b_j|^2,$$

$$\Delta = \Delta(a,b,c) = \rho^{-1}\partial_{\bar{z}}(g)^{-1/2}\rho^2\partial_z\rho^{-1}$$

is the Laplace operator in the instanton field (3), $g = \det(g_{\mu\nu})$, $g_{\mu\nu}$ is the metric on the sphere, det′ denotes the regularized determinant (the diverging part of the determinant is cancelled by the renormalization of the coupling constant). We find next the variation of $\ln \det' \Delta(a,b,c)$ when the parameters of the instanton are varied. We use the fact that the variation of $\mathrm{Sp}\exp(-t\Delta)$ following an infinite similarly small change of the instanton parameters can be written in the form $t[dV(t)/dt]$, where

$$V(t) = \mathrm{Sp}\,(\delta\,(\ln\rho)\,(\exp(-t\Delta) - \exp(-t\tilde{\Delta}))),$$

$$\tilde{\Delta} = g^{-1/2}\rho\,\partial_z\,\rho^{-2}\,\partial_{\bar z}\,\rho.$$

It follows therefore that the variation of $\ln \det_\epsilon \Delta = \int_\epsilon^\infty t^{-1}\mathrm{Sp}\exp(-t\Delta)\,dt$ is equal to $V(\epsilon) - V(\infty)$. This makes it possible to represent the variation of $\ln\det'\Delta$ in the form

$$\frac{1}{\pi}\int\delta\,(\ln\rho)\,\partial_z\partial_z\,(4\ln\rho - \ln\sqrt{g})\,i\,dz\,d\bar z + 2\delta\ln\det(M_{jk}) \tag{6}$$

[$\det_\epsilon\Delta$ can be regarded as a determinant cutoff with respect to the proper time, $\ln\det'$ is the finite part of $\ln\det_\epsilon\Delta$ as $\epsilon\to 0$, the asymptotic form of $V(\epsilon)$ as $\epsilon\to 0$ is calculated by a quasiclassical method, and $V(\infty)$ is expressed in terms of the zeroth modes $\phi_k = \rho^{-1}z^k$ of the operator Δ]. Calculating the integral in (6), we obtain for instanton dimensions that are small compared with the radius R of the sphere the variation of $\ln\det'\Delta - 2\ln\det M_{jk}$ in the form

$$4\delta\ln(|c|^2+1) + 8q\delta\ln|c| + 4\delta\ln\Big(\prod_{j,k}|a_j - b_k|^2\Big).$$

This enables us to obtain $(\det'\Delta)^{-1/2}(\det M_{jk})$ accurate to a constant factor, and leads to the formula indicated above for the instanton contribution. In the case when the instanton can be regarded as a superposition of remote instantons with unity topological charge, the measure $d\mu$ decays in the product of the factors corresponding to this instanton; this makes it possible to calculate the constant factor that has been left undetermined.

[1]A.A. Belavin and A.M. Polyakov, Pis'ma Zh. Eksp. Teor. Fiz. **22**, 503 (1975) [JETP Lett. **22**, 245 (1975)].
[2]A.M. Polyakov, Zh. Eksp. Teor. Fiz. **68**, 1975 (1975) [Sov. Phy. JETP **41**, 988 (1975)].
[3]A. Jevicki, Nucl. Phys. B **127**, 125 (1977).
[4]A. Schwarz, Lett. Math. Phys. **2**, 201 (1978).
[5]I. Frolov and A. Schwarz, Phys. Lett. B (in press).