Dispersion representation for the Lagrange function of an intense field

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The spectral representation is obtained for the exact Lagrange function of a constant electromagnetic field.

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The structure of quantum electrodynamics (QED) at short distances is determined usually by the behavior of the photon propagator at large values of the momentum squared.[1,2] The renormalizability connects the asymptotic properties of the propagator with the behavior of the Gell-Mann–Low function, by incorporating the principal properties of QED. Owing to the universality of the electromagnetic interaction, the radiation action of the electrons via the quantized electromagnetic field can be introduced in QED by functionally differentiating, with respect to the external field, amplitudes that take into account only the electron interaction with the external field (see, e.g., Ref. 3). Therefore the dependence of the quantum-electrodynamic quantities on the external field is of fundamental interest. It is shown in Refs. 4–6 that the exact Lagrangian of the electromagnetic field in the region of a strong field contains the same information on QED as the exact photon propagator at large values of the momentum squared, and determines the corresponding Gell-Mann–Low function.

The study of the properties of the photon propagator is made easy by the fact that we have for this propagator the Kallen–Lehmann dispersion representation.[7] In this paper we obtain an analogous representation for the Lagrangian $\mathscr{L}$ of a constant field; the spectral function in this representation is the imaginary part of the Lagrangian, which is a positive measurable physical quantity, since 2 $\mathrm{Im}\mathscr{L}$ is the probability of pair and photon production by the field per unit volume and per unit time. This representation is

$$\mathscr{L}=\frac{\epsilon^2-\eta^2}{2}-\frac{\alpha\epsilon^2}{\pi}\int_0^\infty\frac{du\,u\,g\left(\frac{u}{e\epsilon},\frac{\eta}{\epsilon}\right)}{u^2-m^4+i\delta}+\frac{\alpha\eta^2}{\pi}\int_0^\infty\frac{du\,u\,g\left(\frac{u}{e\eta},\frac{\epsilon}{\eta}\right)}{u^2+m^4}. \quad (1)$$

Here ϵ and η are the electric and magnetic fields in the system where they are parallel, $(\alpha\epsilon^2/2)g(m^2/e\epsilon,\eta/\epsilon)=\mathrm{Im}\mathscr{L}$ is the imaginary part of the Lagrange function and differs from zero only at $\epsilon\neq 0$. Both $\mathscr{L}$, and g, depend also on the fine-structure constant α. In the second integral, g is connected with $\mathrm{Im}\mathscr{L}$ by means of the permutation $\epsilon\rightleftarrows\eta$.

The derivation of the dispersion relation (1) is based on the existence and properties of the proper-time representation

$$\mathscr{L} - \mathscr{L}^{(0)} = \frac{1}{8\pi^2}\int_0^\infty \frac{dx\, e^{-im^2x}}{x^3} f(m^2x, eFx) \tag{2}$$

for the nonlinear correction to the Lagrangian function $\mathscr{L}^{(0)}$ of the Maxwellian field. For the Heisengberg–Euler correction[8] of order α

$$f^{(1)} = \frac{e\eta x e\epsilon x}{\operatorname{tg} e\eta x \operatorname{th} e\epsilon x} - 1 + \frac{e^2(\eta^2 - \epsilon^2)x^2}{3} = e^{-L}S - 1 + \frac{(eFx)^2}{6} \tag{3}$$

and does not depend on the mass variable m^2x. For the correction $\sim\alpha^2$, obtained by one of us,[4]

$$f^{(2)} = \frac{\alpha}{2\pi}\int_0^1 d\xi \left\{ \frac{e^{-L-L'}}{\xi^2(1-\xi)^2}\left(-2iz(SS' + PP')x^{-1}I_0 - \frac{1}{2}I\right) + \frac{2iz+1}{\xi(1-\xi)} + \frac{(eFx)^2}{6}\left(iz\left(5 - \frac{2}{\xi(1-\xi)}\right) + \frac{5}{2}\right) + \left(\frac{1}{2\xi(1-\xi)} - \ln i\gamma z + \frac{5}{6}\right)\left(4iz + 2 - x\frac{\partial}{\partial x}\right) f^{(1)} \right\} \tag{4}$$

and depends on $m^2x \equiv z$ linearly and logarithmically; the symbols L, S, etc. are defined in Ref. 4, with $s = (1-\xi)x$, $s' = \xi x$. The function $f^{(2)}$ follows from formula (50) of Ref. 4 after the latter is symmetrized with respect to s and s'. The structure of the functions $f^{(1)}$, $f^{(2)}$ in scalar electrodynamics is analogous, see Refs. 9 and 6.

As seen from (3) and (4), the function $f = f^{(1)} + f^{(2)}$ has the following properties: 1) f is analytic in $z = m^2x$ in the lower half of the complex z plane and is here a slowly growing function of $iz \ln iz$. The same applies to m^2 if x is real and positive. 2) Because of the symmetry $\eta \rightleftarrows -i\epsilon$, the singularities of f with respect to the field variable eFx are arranged symmetrically in the complex x plane on the real and imaginary axes (magnetic and electric singularities), and the nearest of them at the points $x = \pm\pi/e\eta$, $\pm i\pi/e\epsilon$ serve as the starting points of the corresponding logarithmic cuts. At the point $x = 0$, f has a logarithmic singularity with respect to the variable m^2x with a cut taken along the half-axis $\arg(m^2x) = \pi/2$. The contour of integration in (2) can be drawn in the sector $-\pi/2 < \arg x < 0$. 3) f is real and analytic on the section $(0, -\pi i/e\epsilon)$ of the imaginary x axis, and therefore satisfies the Riemann–Schwarz symmetry principle

$$f(m^2 x,\ eFx) = f^*(-m^2 x^*,\ -eFx^*) \tag{5}$$

for the points x and $-x^*$ which are symmetrical about the imaginary axis. 4) f is invariant with respect to the reversal of the sign of the field, $F \to -F$, to the reflection $\epsilon \to -\epsilon$, $\eta \to \eta$, and the transformation $\eta \rightleftarrows -i\epsilon$.

By virtue of the property 1, the integral (2) as a function of m^2 has Laplace-transformation properties, i.e., it is an analytic function in the lower half of the complex m^2 plane and tends to zero as $|m^2| \to \infty$. It is therefore possible to apply to it the Cauchy formula, which leads, together with the properties 2–4, to the dispersion representation (1).

The described properties of the representation (2), and hence of representation (1), should be valid in all orders in α, since they reflect the general physical properties of the Lagrange function. Thus, the existence of the Laplace representation (2) itself with respect to the proper time x is due to the causality of the propagation functions, as a result of which $\mathscr{L}$ is analytic in m^2 in the region $\mathrm{Im}\, m^2 < 0$. The existence on the imaginary x axis of a segment $(0, -i\pi/e\epsilon)$ where f is analytic and real is due to the fact that $\mathscr{L}$ is independent of radiative corrections in the weak-field limit: $\mathrm{Re}\mathscr{L}$ should be Maxwellian, and $\mathrm{Im}\mathscr{L}$ should have an essential singularity

$$(e\epsilon)^2 f_0\left(a,\ \frac{\eta}{\epsilon}\right) \exp\left(-\frac{\pi m^2}{e\epsilon}\right), \tag{6}$$

determined by the physical mass of the electron.[4,6] From this and from property 4 it follows that near $x = 0$ the function $f(z,eFx)$ has a structure of the form $(eFx)^4$ (a polynomial of iz and $\ln iz$), so that in the case of a weak field the deviation of $\mathrm{Re}\mathscr{L}$ from a Maxwellian function is of fourth order in the field. Similarly, in a weak field $\mathrm{Im}\mathscr{L}$ behaves like (6), since the dependence on m^2, which is contained in f, drops out of $\mathrm{Im}\mathscr{L}$ effectively as $e\epsilon m^{-2} \to 0$, owing to the cancellation of the contributions of the pole and of the logarithmic branch point with respect to variable eFx at the point $x = -i\pi/e\epsilon$.[4,6] Finally, the Riemann–Schwarz principle, which connect f for two directions of the proper time, is equivalent to the QED symmetry under time reversal.

The obtained representation (1) has the symmetry $\eta \rightleftarrows -i\epsilon$. In the asymmetric strong-field limit, when ϵ/η or $\eta/\epsilon \to 0$, we obtain from (1) the connection $\mathscr{L} = \mathscr{L}^{(0)} Z_3$ with the constant

$$Z_3 = 1 - \frac{2a}{\pi} \int_0^\infty \frac{dx}{x}\, g(x,\ 0) < 1, \tag{7}$$

i.e., the exact Lagrange function is Maxwellian. In finite QED, Z_3 should be finite, positive, and independent of the value of the second argument of the function g, since the limiting interaction constant $\alpha_* = Z_3^{-1}\alpha$ must not depend on the method whereby the field tends to infinity.[5,6] In other words, in finite QED the exact Lagrange function should be Maxwellian in the strong-field limit regardless of the ratio of the electric and magnetic fields.

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