Integrability of supersymmetrical generalizations of classical chiral models in two-dimensional space-time

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Supersymmetrical generators of two-dimensional integrable models of field theory give us examples of interacting fields to which the method of the inverse problem is also applicable. This circumstance was first pointed out by Ferrara *et al.*,[1] who advanced the hypothesis that the supersymmetrical generalization of the sine–Gordon equation has an infinite sequence of integrals of motion. The proof of this hypothesis is given in recent preprints,[2,3] where the "*L-A* pairs" needed to use the inverse-problem

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method were obtained for the first time, and recurrence formulas were obtained for the conservation laws.

It is shown in Ref. 4 that all the presently known integrable models of field theory are gauge-equivalent models of the principal chiral fields. We hope that a similar situation will be observed also for supersymmetrical generalizations of this model. We investigate here therefore supersymmetrical generalizations of the model of the principal chiral fields–superchiral fields–and calculate soliton solutions for them.

Following Whitten,[5] we discuss the supersymmetrical model in the language of superspace in which each point has the usual coordinates x_μ ($\mu = 0,1$) and additional anticommuting coordinates θ_α ($\alpha = 1,2$) that form a Majorana spinor. We define the Dirac γ matrices and $\overline{\psi}$ in the following manner:

$$\gamma^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \gamma^5 = \gamma^0\gamma^1, \quad \overline{\psi} = \psi^+\gamma^0.$$

We take a classical superfield $\hat{\Phi}(x,\theta)$ to mean a field that takes on a value either on an even component G'' of Grassmann algebra, or on an odd component G' (see Ref. 6 concerning Grassmann algebra). Expanding $\hat{\Phi}$ in terms of θ, we get

$$\hat{\Phi} = \Phi + i\theta_2\psi_1 - i\theta_1\psi_2 + i\theta_1\theta_2 F,$$

and furthermore, if $\hat{\Phi}\in G'$, then $\Phi,F\in G'$, $\psi_{1,2}\in G''$; if $\hat{\Phi}\in G''$, then $\Phi,F\in G''$, $\psi_{1,2}\in G'$. Assume that at each point of superspace there is specified an element $\hat{g}(\xi,\eta,\theta_1\theta_2)$ of the group[1)] $SU(N)$, i.e., a matrix $\hat{g}_{\alpha\beta}(\alpha,\beta = 1,2...N)$ that has an expansion

$$\hat{g} = (I + i\theta_2\Lambda_1 - i\theta_1\Lambda_2 + i\theta_1\theta_2 F)g. \tag{1}$$

and satisfies the unitarity condition

$$\hat{g}\hat{g}^+ = \hat{g}^+\hat{g} = I \tag{2}$$

or, in terms of the components,

$$gg^+ = g^+g = I, \quad \Lambda_{1,2} + \Lambda^+_{1,2} = 0, \quad F + F^+ - i\Lambda_1\Lambda^+_2 + i\Lambda_2\Lambda^+_1 = 0. \tag{3}$$

The supersymmetry of the equations for $\hat{g}$ means invariance of these equations to the supertransformation $\delta\hat{g} = (\bar{\epsilon}Q)\hat{g}$ brought about by the generator Q of the supersymmetry group

$$Q = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} = \begin{pmatrix} \partial/\partial\theta_2 + i\theta_2\partial_\xi \\ -\partial/\partial\theta_1 - i\theta_1\partial_\eta \end{pmatrix}$$

ϵ is an infinitesimally small Majorana spinor. We introduce covariant derivatives that anticommute with Q_α:

$$D = \begin{pmatrix} D_1 \\ D_2 \end{pmatrix} = \begin{pmatrix} \partial/\partial\theta_2 - i\theta_2\partial_\xi \\ -\partial/\partial\theta_1 + i\theta_1\partial_\eta \end{pmatrix}.$$

The action for the superchiral field

$$S = \int d\theta_2 d\theta_1 d\xi d\eta \ \mathrm{Sp}(D_1\hat{g}\, D_2\hat{g}^+) \tag{4}$$

leads, when the condition (2) is taken into account, to the equation

$$D_2(D_1\hat{g}\hat{g}^+) - D_1(D_2\hat{g}\hat{g}^+) = 0 \tag{5}$$

or, taking into account expansion (1)

$$2i\Lambda_{2\xi} - [ig_\xi g^+, \Lambda_2)] + \frac{1}{2}[\Lambda_2 \Lambda_1^2] = 0, \tag{6}$$

$$2i\Lambda_{1\eta} - [ig_\eta g^+, \Lambda_1)] + \frac{1}{2}[\Lambda_1 \Lambda_2^2] = 0, \tag{7}$$

$$(g_\xi g^+ - i\Lambda_1^2)_\eta + (g_\eta g^+ - i\Lambda_2^2)_\xi = 0. \tag{8}$$

We note that in the case of 2×2 matrices the system (6)–(8) coincides with the supersymmetrical generalization of the σ model.[6]

We introduce the supercurrents $\hat{u}\in G'$ and $\hat{v}\in G'$:

$$\hat{u} = D_1\hat{g}\hat{g}^+, \qquad \hat{v} = D_2\hat{g}\hat{g}^+. \tag{9}$$

From (5) and condition (2) it follows that they satisfy the system of equations

$$D_2\hat{u} - D_1\hat{v} = 0, \quad D_2\hat{u} + D_1\hat{v} - \{\hat{u}\hat{v}\} = 0. \tag{10}$$

The system (10) can be regarded as the condition for the existence of a simultaneous solution $\hat{\psi}(\xi,\eta,\theta_1,\theta_2,\lambda)$ of the pair of linear problems for all values of the complex parameter λ:

$$D_1\hat{\psi} = (1+\lambda)^{-1}\hat{u}\hat{\psi}, \quad D_2\hat{\psi} = (1-\lambda)^{-1}\hat{v}\hat{\psi}, \tag{11}$$

where $\hat{g} = \hat{\psi}(\xi,\eta,\theta_1,\theta_2,0)$ satisfies the system (5).

An important question is that of the "classical vacuum" of superchiral fields. In the theory of ordinary chiral fields on the $SU(N)$ group the vacuum solution was naturally chosen to be the diagonal matrices $u^0(\xi) = ig_{0\xi}g_0^+$, $v^0(\eta) = ig_{0\eta}g_0^+$ which are not equal to zero.[4] The soliton solutions of the equations for u and v emerged asymptotically from this vacuum. The internal arrangement of the solitons and the very fact of their existence depends on which solution u^0,v^0 is taken as the vacuum solution. Proceeding similarly, we start here likewise from diagonal matrices u_0 and v_0 satisfying the equation (10):

$$\hat{u}_0 = i\chi_1(\xi) - \theta_2 u^0(\xi), \quad \hat{v}_0 = i\chi_2(\eta) + \theta_1 v^0(\eta). \tag{12}$$

We calculate the wave function $\hat{\psi}_0$ as a solution of Eq. (11)

$$\hat{\psi}_0 = \left(I - \frac{\theta_2\chi_1(\xi)}{1+\lambda} + \frac{\theta_1\chi_2(\eta)}{1-\lambda} + \frac{\theta_1\theta_2\chi_1(\xi)\chi_2(\eta)}{1-\lambda^2}\right)$$

$$\times \exp\left(\frac{i\int u^0(\xi)\,d\xi}{1+\lambda} + \frac{i\int v^0(\eta)\,d\eta}{1-\lambda}\right).$$

The procedure of obtaining the soliton solution is a trivial generalization of the results of Ref. 4 to the case of superspace. We represent the function $\hat{\psi}$ in the form

$$\hat{\psi}=\left(I-\frac{\lambda_0-\bar{\lambda}_0}{\lambda-\bar{\lambda}_0}\hat{P}\right)\hat{\psi}_0 ,$$

where $\hat{P}$ is the projection operator ($\hat{P}^2=\hat{P}$). It follows from (11) that $\hat{P}$ satisfies the system of equations

$$(1-\hat{P})(D_1-(1+\bar{\lambda}_0)^{-1}\hat{u}_0)\hat{P}=0,\ (1-\hat{P})(D_2-(1-\bar{\lambda}_0)^{-1}\hat{v}_0)\hat{P}=0$$

whose general solution is the operator $\hat{P}$ that projects into a space spanning the basis set

$$|\hat{e}^i(\xi,\eta,\theta_1,\theta_2)>=\hat{\psi}_0(\xi,\eta,\theta_1,\theta_2,\bar{\lambda}_0)|e^i_0>$$

where $|e^i_0\rangle$, $i=1,...,k$, is some linearly independent set k of the vectors. We construct $\hat{p}^i=|\hat{e}^i\rangle\langle\hat{e}^i|/\langle\hat{e}^i|\hat{e}^i\rangle$, and then if $k=1$ we get $\hat{P}=\hat{P}^1$ and if $k=2$, then

$$\hat{P}=\frac{\hat{p}^1+\hat{p}^2-\hat{p}^1\hat{p}^2-\hat{p}^2\hat{p}^1}{1-\mathrm{Sp}(\hat{p}^1\hat{p}^2)},$$

etc. Substituting the calculated projector $\hat{P}$ in (13) at $\lambda=0$ we obtain the soliton of the field $\hat{g}$.

The soliton solution is a single wave that goes over at an exponential rate to the vacuum solution. It is in effect the solution of the nonlinear equation for the usual chiral model and a sequence of linear equations. This can be easily verified by expanding the field in the basis of the Grassmann algebra. The method proposed here makes it possible to obtain the solution in closed and compact form even in the case when the number of generators of the Grassmann algebra is infinite.

We note in conclusion that the conditions (11), expressed in terms of the components, has a mixed differential-algebraic character which was not encountered earlier in the inverse-problem method. The use of compatibility conditions of this kind and the resorting to an expanding superalgebra $\theta^i_\alpha(i=1,...,m)$ may be useful even in the case when the fields are ordinary functions (and not elements of Grassmann algebra). The integrals of motion for the model considered here can be obtained in the traditional manner, the generalization of which to superfields is trivial.[2,3]

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[1)]We have confined ourselves for brevity to the $SU(N)$ group, since chiral fields on the other groups (SO(N), Sp($2N$)) can be generalized to the supersymmetry case in analogous manner.

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