Three-dimensional localization of helicons

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A nonlinear three-dimensional Schrodinger equation describing the evolution of the amplitude of a helicon packet propagating along a constant magnetic field is obtained. It is shown that dispersion and nonlinear effects lead to three-dimensional localization of the helicons.

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Helicons are electromagnetic plasma oscillations in the frequency range $\omega_{Bi} \ll \omega \ll \omega_{Be}, \omega_{Pe}$ and are frequently observed in the form of noise in radio signals.[1] This noise is repetitive in form, and this indicates that the helicons have a tendency to become self-localized and to form stationary soliton packets. Helicons can be easily excited in a plasma by beams of high-energy electrons propagating along or across the magnetic field,[2] so that such waves can possibly build up in a laboratory plasma, including thermonuclear research installations. The self-localization ability is an important property, since it makes possible the accumulation of a large amount of wave energy, even when the instability region and the growth rate are small.

We assume that the helicon packet has a principal wave number k_0 directed along a constant magnetic field ($\mathbf{k}_0 \| \mathbf{B}_0 \| \mathbf{z}$), and that this number can be much larger than the width of the packet in the wave-number space. We consider wave packets that have sufficiently small dimensions in coordinate space. The decay of a helicon wave into two helicons with lower frequency can therefore be neglected, since the group velocities of these waves are different and they cannot interact with each other for sufficiently long time in a three-dimensional packet. The main nonlinear effect in our case is the action of the high-frequency pressure of the packet on the plasma, which leads to the formation of density walls and of a magnetic field in the region of localization of the packet, and which moves together with the packet. As will be seen below, these walls hinder the spreading of the packet. To find the parameters of these walls, we use the magnetohydrodynamic equations averaged over the fundamental frequency of the packet.

Assuming that there are two types of motion in the system, high-frequency and low-frequency, we represent the magnetic field and the particle density and velocity in the form

$$\mathbf{B} = \mathbf{B}_o + \mathbf{B}_1 + \mathbf{B}_2; \quad n = n_o + n_1 + n_2; \quad \mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2, \tag{1}$$

where $\mathbf{B}_1, n_1, \mathbf{v}_1$ and $\mathbf{B}_2, n_2, \mathbf{v}_2$ are respectively the high- and low-frequency perturbations.

After substituting the representations (1) in the magnetohydrodynamic equations and averaging over the fundamental frequency of the packet we get

$$m_i n_o \frac{\partial \mathbf{v}_2}{\partial t} + \frac{B_o}{4\pi} \nabla B_{2z} - \frac{B_o}{4\pi} \frac{\partial \mathbf{B}_2}{\partial z} = - m_i n_o < (\mathbf{v}_1 \nabla) \mathbf{v}_1 >$$

$$- \nabla < \frac{B_1^2}{8\pi} > + \frac{1}{4\pi} < (\mathbf{B}_1 \nabla) \mathbf{B}_1 > \equiv < \mathbf{C} >, \tag{2}$$

$$\frac{\partial n_2}{\partial t} + n_o \operatorname{div} \mathbf{v}_2 = - < \operatorname{div} n_1 \mathbf{v}_1 > \equiv < \psi >, \tag{3}$$

$$\frac{\partial \mathbf{B}_2}{\partial t} + \mathbf{B}_o \operatorname{div} \mathbf{v}_2 - B_o \frac{\partial \mathbf{v}_2}{\partial z} = - < \mathbf{B}_1 \operatorname{div} \mathbf{v}_1 > + < (\mathbf{v}_1 \nabla) \mathbf{v}_1 >$$

$$- < (\mathbf{v}_1 \nabla) \mathbf{B}_1 > \equiv < \mathbf{D} > . \tag{4}$$

The angle brackets denote here averaging over the fast oscillations, and $\langle \mathbf{C} \rangle \langle \psi \rangle$, $\langle \mathbf{D} \rangle$ are the contributions of the high-frequency quantities to the corresponding low-frequency equations. Combining the z-component of Eqs. (2) and (4), we can obtain an expression for $\partial (\operatorname{div} \mathbf{v}_2)/\partial t$. Next, taking the derivative of (4) with respect to time and substituting in it the expression for $d(\operatorname{div} \mathbf{v}_2)/dt$, we obtain

$$\frac{\partial^2}{\partial t^2}\left(\frac{n_2}{n_o} - \frac{B_{2z}}{B_o}\right) = - \frac{1}{m_i n_o} \cdot \frac{\partial < C_z >}{\partial z} - \frac{1}{B_o} \frac{\partial < D_z >}{\partial t} + \frac{\partial < \psi >}{\partial t} \cdot \tag{5}$$

The equation for the high-frequency oscillations of the magnetic field is obtained from the dispersion relation of the helicons:

$$\omega = \frac{c^2\omega_{Be}}{\omega_{Pe}^2} k k_z \approx \frac{c^2\omega_{Be}}{\omega_{Pe}^2} (k_z^2 + \tfrac{1}{2} k_\perp^2); \quad k_\perp^2 << k_z^2 . \tag{6}$$

It takes the form

$$i\frac{\partial \mathbf{B}_1}{\partial t} + \frac{c^2\omega_{Be}}{\omega_{Pe}^2}\left(\frac{\partial^2 \mathbf{B}_1}{\partial z^2} + \frac{1}{2}\Delta_\perp \mathbf{B}_1\right) + \frac{\partial^2}{\partial z^2}\left\{\left(\frac{B_{2z}}{B_0} - \frac{n_2}{n_0}\right)\mathbf{B}_1\right\} = 0 \tag{7}$$

The notation here and below is standard. We separate from $\mathbf{B}_1$ the principal number, i.e., we represent it in the form

$$\mathbf{B}_1 = \sqrt{\frac{4\omega_0}{\omega_{Bi}}} B_0 \left\{\mathbf{b}\exp(ik_0 z - i\omega_0 t) + \text{c.c.}\right\}; \quad \omega_0 = \frac{c^2\omega_{Be}}{\omega_{Pe}^2} k_0^2 . \tag{8}$$

Here ω_0 is the fundamental frequency of the packet and $\mathbf{b}$ is a dimensionless amplitude that depends little on z, r_1, or t.

Taking into account the representation (8) and the fact that in the linear approximation the following conditions are satisfied in a helicon propagating along the magnetic field

$$n_1 = 0; \quad v_{1z} = 0; \quad B_{1z} = 0$$

we can neglect the last two terms in (5), and $\langle C_z \rangle$ takes the form

$$\langle C_z \rangle = -\frac{B_0^2}{\pi}\frac{\omega_0}{\omega_{Bi}}\frac{\partial |b|^2}{\partial z} . \tag{9}$$

Introducing the group velocity of the packet $V_9 = 2c^2\omega_{Be}k_0/\omega_{Pe}^2$, which corresponds to the wave number k_0, we can assume that all the numerically varying quantities depend on the coordinates and the time in the following manner:

$$b = b(z - V_9 t, r_\perp, t) = b(\xi, r_\perp, t), \tag{10}$$

where the dependence on the last argument is much weaker than the dependence on the time, which enters in the first argument. Taking (9), (10), and the foregoing into account, relation (5) becomes

$$\frac{n_2}{n_0} - \frac{B_{2z}}{B_0} = |b|^2. \tag{11}$$

With allowance for (8), (9), (11), we get from (7) ultimately

$$i\frac{\partial b}{\partial t} + \frac{c^2\omega_{Be}}{\omega_{Pe}^2}\left(\frac{\partial^2 b}{\partial \xi^2} + \frac{1}{2}\Delta_\perp b\right) + \omega_0 |b|^2 b = 0. \tag{12}$$

We have thus obtained the equation for slow evolution of the amplitude of a helicon wave packet with a carrier wave number k_0 and a carrier frequency ω_0, in a coordinate system that moves with the group velocity of the wave. It is seen from (12) that this evolution is due to dispersion and to high-frequency pressure. Equation (12) is the well known nonlinear Schrodinger equation. It was shown in Ref. 3 that three-dimensional soliton solutions of (12) are unstable, but since the growth rate of this instability is small (it is proportional to the square of the soliton amplitude), the helicon turbulence will exist mainly in the form of a cluster of such solitons. It is this which explains the frequent recurrence in the form of helicon noise ("whistlers") in the ionosphere.

We now obtain the equation for the soliton solution (12). To this end we put $b = Af(\rho)\exp(iA^2\tau)$. Here ρ, τ, and A are the dimensionless radius, time, and amplitude, respectively, $\rho = Ak_0(\xi^2 + 2r_\perp^2)^{1/2}$; $\tau = \omega_0 t$. For $f(\rho)$ we get

$$\frac{1}{\rho^2}\frac{\partial}{\partial\rho}\left(\rho^2\frac{\partial f}{\partial\rho}\right) = f - f^3. \tag{13}$$

This equation has a soliton solution with amplitude and characteristic dimensions of the order of unity. From this we obtain a connection that might be verified in experiment between the parameters of the helicon soliton. The duration of the passage is of the order of

$$\left(\frac{B_1}{B_0}\sqrt{\frac{\omega_{Bi}}{\omega_0}}\,\omega_{Bi}\right)^{-1}$$

i.e., it is inversely proportional to the amplitude divided by the square root of the carrier frequency of the packet.

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