Spivak for continuous interest in the work, and also A.I. Afonin, A.G. Roshchin, A.Yu. Kulikov, and S.I. Kuznetsov for taking part in developing the apparatus and performing the measurements. The authors are grateful to the crew of the IRT-M reactor of the Kurchatov Atomic Energy Institute headed by V.P. Chernyshevich.

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WKB METHOD FOR A STRONG COULOMB FIELD

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Submitted 8 January 1971
ZhETF Pis. Red. 13, No. 7, 359 - 362 (5 April 1971)

As is well known, the WKB method for the Schrodinger equation with a Coulumb potential leads to terms that coincide with the exact solution for all quantum numbers. The same holds true for the Dirac equation at $Z < 137$. On this basis it has been proposed (and confirmed by calculation) that for $Z > 137$ the electronic terms are well described by the WKB approximation for all quantum numbers.

Greatest interest attaches to the behavior of the lowest level $1s_{1/2}$, which is the first to reach the boundary of the lower continuum with increasing Z. To exclude striking the center, the nucleus was assumed to be finite (radius $R \ll 1$) and having a constant potential. The WKB method was used to find the term $1s_{1/2}$ in a field $V = -\alpha/r$ at $R > R$ and $V = -\alpha/R$ at $r < R$. Here $\alpha = Z/137$. The Dirac equation for the radial function $G(r)$ was transformed to the self-adjoint form $\phi'' + k^2\phi = 0$ by means of the substitution $G(r) = [1 + \varepsilon - V(r)]^{1/2}\phi(r)$, with

$$k^2 = \epsilon^2 - 1 + 2\epsilon a/r + a^2/r^2 - \frac{3}{4} r^{-2}[1 + r(1+\epsilon)/a]^{-2} \qquad (1)$$

when $r > R$ and $k^2 = (\varepsilon + \alpha/R)^{\pm} - 1$ when $r < R$. A system with units $h = m = c = 1$ is used throughout. In order to satisfy the WKB condition at small values of r, the Langer correction $-1/4r^{-2}$ [1] was added to (1). We assume first $\varepsilon = -1$ and find the connection between Z_{cr} and R. The effective potential for this case is shown in Fig. 1. The Bohr quantization rule in a potential with a single vertical return wall has the form [2] $\int k dr = [n + (3/4)]\pi$. Applying this relation to the term $1s_{1/2}$, we obtain (assuming $R \ll 1$, which is actually always the case)

$$R = 2g_{cr}^2 \alpha_{cr}^{-1} \exp\left[-2 - \left(\frac{3}{4}\pi - \alpha_{cr}\right)/g_{cr}\right], \qquad (2)$$

where

$$g_{cr} = \sqrt{\alpha_{cr}^2 - 1}.$$

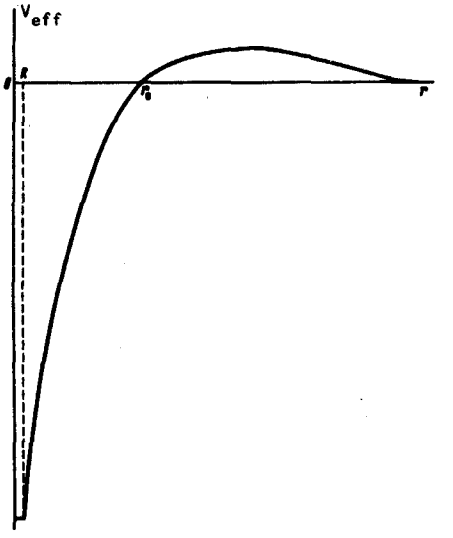
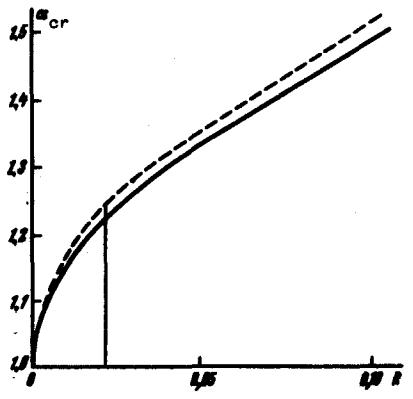


Fig. 1. Effective potential for $\alpha = \alpha_{cr}$.



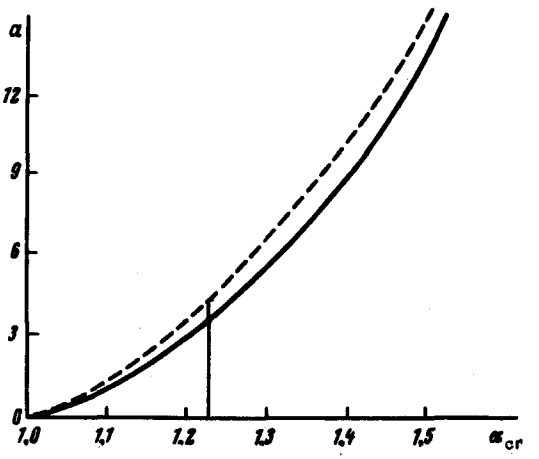


Fig. 2. α_{cr} vs. the nuclear radius R. Solid line - WKB calculation, dashed - exact numerical calculation [3]. Vertical line - real critical radius.

Fig. 3. Coefficient a from the formula $W \sim \exp[-\sqrt{a/(\alpha - \alpha_{cr})}]$ for the probability of passing through the barrier at $\alpha > \alpha_{cr}$ vs. α_{cr}. Solid and dashed lines - the same as in Fig. 2. Vertical line - real value of α_{cr}.

The solid line in Fig. 2 shows the dependence of α_{cr} on R, obtained from formula (2). The dashed curve is the result of the exact numerical calculation [3]. Putting $R = 1.1A^{1/3}F$ and $A = 2.5\,Z$, this yields $Z_{cr} = 170$.

We then examine the behavior of the $1s_{1/2}$ level in the vicinity of the point $\alpha = \alpha_{cr}$. For $\alpha < \alpha_{cr}$ we obtain from the Bohr quantization rule

$$\epsilon = -1 + C_1(\alpha_{cr} - \alpha),$$

where

$$2C_1 = \left(\frac{3}{4}\pi\alpha_{cr}^2 - \alpha_{cr} + g_{cr}\alpha_{cr}^2 + g_{cr}\right)\alpha_{cr}g_{cr}^{-3}\left(\frac{7}{8} + \frac{g_{cr}^2}{3}\right)^{-1}. \qquad (3)$$

The same formula holds also for $\alpha > \alpha_{cr}$, with the only modification that in this case the level becomes quasistationary [3]. The probability of passing through the barrier at $\alpha > \alpha_{cr}$, calculated in the WKB approximation, is equal to $\exp[-\sqrt{a/(\alpha - \alpha_{cr})}]$, where

$$a = 2\alpha_{cr}^2\pi^2/C_1. \qquad (4)$$

The solid line of Fig. 3 shows a plot of α against α_{cr} in accordance with formulas (3) and (4). The dashed curve is the result of the exact numerical calculation [3].

We then calculate within the framework of the WKB method the average distance r between the electron and the center. It turns out that it is important to take into account not only the classically accessible region, but also the inaccessible one. For the case $Z = Z_{cr} = 170$, the value of r turned out to be 0.24 (as against 0.29 from the exact calculation [3]), i.e., larger than the width of the classically accessible region r_0 (see Fig. 1), which equals 0.20 at $Z_{cr} = 170$. This means that in the region $Z = Z_{cr}$ the electron in the $1s_{1/2}$ state spends the greater part of the time in classically inaccessible region.

The agreement between the WKB approximation and the exact calculations for terms with large quantum numbers turned out to be, naturally, even better than for the $1s_{1/2}$ term.

I am grateful to A.B. Migdal for valuable advice.

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COLLECTIVE EFFECTS IN DISCRETELY-ELECTRONIC BEAMS

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Submitted 17 February 1971
ZhETF Pis. Red. 13, No. 7, 362 - 365 (5 April 1971)

The purpose of the present communication is to call attention to collective effects that arise as a result of the discrete character of the elementary charge in beams of electrons (or other charged particles) in a vacuum, and make it possible to liken such beams, in certain respects, to pseudocrystalline structures.

Assume that the electrons move in vacuum along linear trajectories in a strong longitudinal magnetic field. At a given beam current I, the average longitudinal distance between neighboring beam electrons is $a = ev/I$, where $e = |e|$ is the elementary charge and v the velocity of the electrons. If the magnetic field is sufficiently strong, so that the transverse dimension of the beam is $d \ll a$, then such a thin beam has the form of a linear discretely-electronic chain recalling certain known models of one-dimensional crystals.

The initial distribution of the longitudinal coordinates of the electrons in the chain, determined by the shot effect of the cathode, by the thermal velocities of the electrons, and by the conditions for the formation of the thin beam, has a random character. Therefore the interval between any two neighboring electrons is $a_i(t) \neq a$. With further linear motion, a tendency for ordering occurs in the chain under the influence of the longitudinal Coulomb-repulsion forces, i.e., a tendency of the potential energy of the chain to become minimal as $a_i(t > \tau) \to a$, where τ is a certain relaxation time. This tendency is resisted by the longitudinal thermal motion of the electrons, thus imposing on the temperature T of the ordered beam the limitation

$$|z/a| \approx [akT/4{,}8e^2]^{1/2} = 0.46\left[\frac{a}{r_0}\frac{kT}{mc^2}\right]^{1/2} \ll 1.$$

Here z is the longitudinal coordinate of the electron, $z = 0$ corresponds to the equilibrium position of the electron in a perfectly ordered chain at $a_i = a$,