N-soliton solutions of the nonlinear Landau–Lifshitz equation

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The Marchenko method is used to find a new class of N-soliton type particular solutions of the nonlinear Landau–Lifshitz equation that describes the dynamics of an anisotropic ferromagnetic medium with allowance for the magnetic–dipole interaction.

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1. The nonlinear dynamics of a ferromagnet with uniaxial anisotropy is described by the Landau-Lifshitz (LL) equation[1]

$$\vec{\mu}_t = \vec{\mu} \times \vec{\mu}_{xx} + \beta(\vec{\mu} \times \mathbf{n})(\vec{\mu}\mathbf{n})^{1)}, \quad (1)$$

where $\vec{\mu}$ is a unit vector in the direction of the vector of the density of the magnetic moment of the medium, **n** is a constant unit vector along the anisotropy axis, and β is a constant of the anisotropic interaction in the crystal. Since we are considering in the present communication the simplest geometry of motions in a medium (the wave propagates along the anisotropy axis), allowance for the magnetic-dipole interaction leads simply to a renormalization of the constant β.

The nonlinear properties of magnetically ordered media were first investigated in Ref. 2, where solutions of Eqs. (1) were obtained in the form of "solitary" waves–magnetic solitons. In Ref. 3, which reports the results of a numerical experimental investigation of the interaction of magnetic solitons, it was shown that Eq. (1) has also a two-soliton solution. The last fact, wherein the solitons pass through each other and remain unchanged in shape and amplitude asymptotically with time, is the classical argument favoring the assumption that the LL equation belongs to the class of fully integrable evolution equations.[2)]

2. We consider the following pair of operators:

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \frac{\partial}{\partial x} - \begin{pmatrix} 0; & \lambda\mu^3 & -\lambda\mu^2 \\ -\lambda\mu^3;\ 0; & 0\ ; & \lambda\mu^1 \\ (\lambda+\beta/\lambda)\mu^2; & -(\lambda+\beta/\lambda)\mu^1; & 0 \end{pmatrix} \tag{2}$$

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \frac{\partial}{\partial x}$$

$$- \begin{pmatrix} 0 & (\lambda^2+\beta)\mu^3 + \lambda\nu^3 & -\lambda^2\mu^2 - \lambda\nu^2 \\ -\lambda^2\mu^3 - \lambda\nu^3 - \beta\mu^3 & 0 & \lambda^2\mu^1 + \lambda\nu^1 \\ (\lambda^2+\beta)\mu^2 + (\lambda+\beta/\lambda)\nu^2; & -(\lambda^2+\beta)\mu^1 - (\lambda+\beta/\lambda)\nu^1; & 0 \end{pmatrix} \tag{2'}$$

where λ is a parameter that does not depend on the coordinates or the time. The operators (2) and (2') commute with each other if the vector $\vec{\mu}$ satisfies the equation (1) investigated by us, and if[3)] $\vec{\nu} = \vec{\mu} \times \vec{\mu}_x$.

The operators (2) and (2') play just as important a role in the investigation of the initial nonlinear Eq. (1) as does the Lax pair in the method of the inverse scattering problem.[6] Here, however, following the Marchenko method,[7] we seek for the system of equations generated by these operators

$$L^{ik} y^k = 0; \quad A^{ik} y^k = 0 \quad (i, k = 1, 2, 3) \tag{3}$$

a simultaneous solution in the form of polynomials in λ:

$$g^* = f = y^1 + i y^2 = \sum_{i=1}^{N} f^i (\lambda)^i\ ; \quad h = y^3 = \sum_{i=0}^{N} h^i (\lambda)^i\ .$$

We obtain here a certain set of algebraic and differential equations for the polynomial coefficients f^i, g^i, and h^i, which are functions of the coordinates and the time. It turns out that

$$f^N=(g^N)^*=\mu^1+i\mu^2\,;\ h^N=\mu^3. \tag{4}$$

Using (4), we close the system of equations for the polynomial coefficients. The obtained autonomous system of differential equations for the polynomial coefficients is compatible, since it satisfies identically the Frobenius condition, $f^i_{xt}=f^i_{tx}$; $h^i_{xt}=h^i_{tx}$. Consequently, the system (3) does in fact have a solution that depends polynomially on λ, and a direct check shows that each polynomial solution of this system defines for us, in accord with (4), a certain solution of the LL equations (1). Instead of investigating the equations for the polynomial coefficients, it is convenient for us to investigate the equations of motion of the zeros z^i and $\bar{z}^i$ of the polynomials f and g, respectively

$$z^k_x=i\,\frac{z^k h(z^k)}{\prod\limits_{l\neq k}(z^k z^l)} \tag{5}$$

$$z^k_t=-i\,\frac{z^k h(z^k)}{\prod\limits_{l\neq k}(z^k z^e)}\left(\Sigma z^l-z^k+\frac{1}{2}K\right) \tag{5'}$$

where K is the sum of the zeros of a, generally speaking, rational function in λ:

$$P(\lambda)=\lambda^2(\lambda^2+\beta)^{-1}h^2+fg, \tag{6}$$

which does not depend on the coordinates and the time.

In turn, the solution of Eq. (1) is determined in terms of z^i and $\bar{z}^i$:

$$(\mathrm{arc\,th}\,(\mu^3))_x=(1/2i)\sum_k(\bar{z}^k-z^k) \tag{7}$$

$$(\mathrm{arc\,th}\,(\mu^3))_t=(-1/2i)\{\sum_{l>k}(\bar{z}^k z^l-z^k z^l)-\frac{1}{2}K\sum_k(\bar{z}^k z^k). \tag{7'}$$

Although the nonlinear but ordinary differential equations (5) and (5′) for the zeros are complicated, standard substitutions lead to their solution, and the entire difficulty lies in principle in the inversion of these substitutions.

3. Among the particular solutions of Eq. (1), which are generated by polynomial solutions for the pseudopotential function y^k, the most interesting are solutions N-soliton type, which correspond to polynomials f, g, and h such that the function $P(\lambda)$ has only multiple roots. It turns out then that essentially different cases arise, corresponding to different signs of the anisotropy [the sign of β corresponds to physically different objects, namely, $\beta>0$ ($\beta<0$) corresponds to a ferromagnet with anisotropy of the easy axes (easy plane) type].

For lack of space, we limit ourselves to the case $\beta>0$.

A) If $P(\lambda)$ has two degenerate roots, we obtain the previously found[2] single-soliton solution[4]

$$\mu^3 = \frac{4\beta - v^2}{2\beta + \sqrt{\beta} v \operatorname{ch}\{(4\beta - v^2)(x - vt)\}}. \tag{8}$$

B) A two-soliton solution of (1) is obtained by using $P(\lambda)$ with four degenerate roots. The asymptotic behavior of this solution is standard for the two-soliton solution: as $t \to -\infty$ the solution constitutes two infinitely separated solitons moving with different velocities v_1 and v_2, so that the distance between them decreases, and as $t \to \infty$ the solution again comprises two infinitely separated solitons that move with the same velocities (and consequently having the same amplitude as prior to the interaction), but the distance between them increases. The initial phase of the interacting solitons changes by an amount

$$\Delta = \ln \frac{8\beta - 2v_1v_2 + 2\sqrt{(4\beta - v_1^2)(4\beta - v_2^2)}}{8\beta - 2v_1v_2 - 2\sqrt{(4\beta - v_1^2)(4\beta - v_2^2)}}. \tag{9}$$

C) Although the increase of the degree of the polynomials f, g, and h greatly complicates the equations of motion (5) and (5′) of the zeros, the change of variable with which the two-soliton solution was obtained can be easily generalized. As a result, a function $P(\lambda)$ having $2N$ degenerate roots will lead to an N-soliton solution, which will be presented in a larger communication.

4. In conclusion, the author takes the opportunity to express deep gratitude to Professor V.A. Marchenko for constant interest in the work and for valuable advice and remarks, and also to E.N. Bratus', V.A. Kozel, V.P. Kotlyarov, K.V. Maslov, V.N. Robuk, A.A. Slutskin, and E.Ya. Khruslov for useful discussions in the course of the discussion of the work.

[1)]Here and below, a subscript denotes the corresponding partial derivative.

[2)]It was shown recently[4,5] that the nonlinear dynamic equation of the isotropic Heisenberg model $\vec{\mu}_t = \vec{\mu} \times \vec{\mu}_{xx}$ is fully integrable. Trakhtajah[5] obtained two-soliton solutions and pointed to the existence of N-soliton solutions. Equation (1), however, is substantially different. It should be noted that Eq. (1) is a more accurate model of the physical processes that occur in a ferromagnet, since it takes into account both the anisotropic and the magnetic-dipole interactions.

[3)]An exposition of the method of finding the operators (2) and (2′) associated with the LL equation (1) will be the subject of a separate communication.

[4)]The solution (8) corresponds to a definite choice of the integration constants. A different choice leads to the single-soliton solutions obtained in Ref. 8. The dependence of the type of solution on the choice of the integration constants means apparently that there exists some transformation that reduces the solution of an equation with one value of β into a solution of an equation with a different value of β, and in particular to a solution of an equation with $\beta = 0$, which, as indicated above, can be solved by the method of the inverse scattering problem.

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