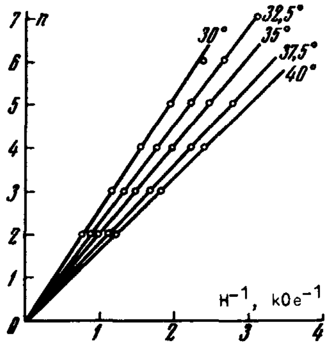

Fig. 3. Dependence of the number of oscillations n on the reciprocal magnetic field for several different directions of the vector H in the plane of the binary and trigonal axes of the crystal. k is parallel to the binary axis. The numbers on the lines indicate the angle between k and H.

$$m_c \bar{v} = \frac{1}{2\pi} \frac{\partial S(p_z)}{\partial p_{z.\mathrm{lim}}} = \frac{e}{ck_z \Delta H^{-1}} \qquad (3)$$

The initial phase of the oscillations, in accordance with condition (1), should be equal to zero. Figure 3 shows the numbers of the oscillations to be functions of the reciprocal magnetic field for several directions of the vector H. It is seen from the figure that the initial phase of all series of oscillations is indeed equal to zero.

In conclusion we are sincerely grateful to E. A. Kaner for useful discussions and also to N. D. Belousov for help with the measurements.

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CASIMIR OPERATORS FOR THE ORTHOGONAL AND SYMPLECTIC GROUPS

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As is well known [1], the name orthogonal group O(n) is given to the group of linear transformations which conserve the quadratic form $\Sigma_{i=1}^{n} (x^i)^2$; analogously, the symplectic group Sp(2n) consists of unitary transformations which conserve the bilinear form $\Sigma_{i=1}^{n} (y^i x^{-i} - y^{-i} x^i)$. The simplest orthogonal groups O(2), O(3), and O(4) have numerous applications in physics; the orthogonal groups of higher order, as well as the symplectic groups, are used in the classification of states in the nuclear-shell model [2]. This frequently raises the problem of finding invariant operators (the so-called Casimir operators) which can be constructed from the generators of the given group. The most important in physics is the quadratic Casimir operator C_2, the eigenvalues of which were obtained by Racah [2]. Explicit expressions for the eigenvalues of the operators C_p with $p > 2$ were never published (with the exception of the operator C_4 for the group Sp(4), see [3]). We present below a solution of this problem in general form.

With a single exception (see (7) below), the Casimir operator C_p of arbitrary order p, for the groups O(n) and Sp(2n), is of the form

$$C_p = \sum_{i_1,\dots i_p} X^{i_1}_{i_2} X^{i_2}_{i_3} \dots X^{i_p}_{i_1} \tag{1}$$

where X^i_j are the generators of the group in question. Let the irreducible representation be specified by the Young tableau $\{f_1, f_2, \dots, f_\nu\}$, where f_i is the number of boxes in the i-th row, $f_1 \geq f_2 \geq \dots \geq f_\nu \geq 0$. The eigenvalue of the operator C_p for this representation will be denoted by $C_p(f_1, \dots, f_\nu)$. Using for its calculation the same method [4] as for the unitary group, we obtain

$$C_p(f_1, f_2, \dots, f_\nu) = \sum_{i,j} (a^p)_{ij} \tag{2}$$

The matrix $\underline{a}$ contained in this expression is given by

$$a_{ij} = (\ell_i + \alpha)\, \delta_{ij} - \theta_{ij} + \beta \frac{1 + \epsilon_i}{2}\, \delta_{i,-j} \tag{3}$$

Here $\ell_i = f_i + r_i$ (for $i > 0$), $\ell_{-i} = -\ell_i$; $\epsilon_i = +1$ when $i > 0$, 0 when $i = 0$, and -1 when $i < 0$; $\theta_{ij} = 1$ when one of the following conditions is satisfied: $0 < i < j$, $i < j < 0$, $i \geq 0 \geq j$ (except $i = j = 0$); $\theta_{ij} = 0$ in all other cases. The quantities α, β, r_i, and also the values assumed by the indices i and j for the different groups are listed in the table.

Group: Cartan notation	Group: Other notation	α	β	r_i	Index i runs through the values
A_{n-1} OU(i)	U(n)	$\frac{n-1}{2}$	0	$\frac{n+1}{2} - i$	1, 2, ..., n
B_n	O(2n + 1)	$n - \frac{1}{2}$	1	$(n + \frac{1}{2})\epsilon_i - i$	1, 2, ..., n, 0, - n, ..., - 2, - 1
C_n	Sp(2n)	n	-1	$(n + 1)\epsilon_i - i$	1, 2, ..., n, - n, ..., - 2, - 1
D_n	O(2n)	n - 1	1	$n\epsilon_i - i$	1, 2, ..., n, - n, ..., - 2, - 1

From (2) and (3) we obtain the explicit for of C_p for p = 2, 3, and 4:

$$C_2 = 2S_2, \quad C_3 = (2\alpha - \beta + 1)S_2, \quad C_4 = 2S_4 - (2\alpha\beta + \beta - 1)S_2 \tag{5}$$

These expressions are valid for any of the groups O(2n + 1), O(2n), and Sp(2n). Here

$$S_2 = \sum_{i=1}^{n} (\ell_i^2 - r_i^2), \qquad S_4 = \sum_{i=1}^{n} (\ell_i^4 - r_i^4). \tag{6}$$

Since O(2n + 1), O(2n), and Sp(2n) are groups of rank n, each contains n independent Casimir operators. It is known [5] that the operators C_p with odd p can be expressed in terms of C_{2q} with $2q < p$. In the case of the groups O(2n + 1) and Sp(2n), the operators $C_2, C_4, \dots, C_{2n}$

form a complete set of independent invariant operators. A special situation arises for the grou O(2n): in order for the eigenvalues of the invariant operators to characterize the irreducible representation uniquely, the operator C_{2n} must be replaced by the operator C'_n:

$$C'_n = \epsilon^{j_1 j_2 \ldots j_n}_{i_1 i_2 \ldots i_n} X^{i_1}_{j_1} X^{i_2}_{j_2} \ldots X^{i_n}_{j_n} \qquad (7)$$

which is analogous to the pseudoscalar $\epsilon_{\mu\nu} u_{\mu\nu} u_{\rho\sigma}$ in the Lorentz group. The eigenvalues of C'_n are:

$$C'_n(f_1, \ldots, f_n) = (-1)^{\frac{n(n-1)}{2}} 2^n n! \ell_1 \ell_2 \ldots \ell_n \qquad (8)$$

In conclusion we note that not all representations of the groups O(2n) and O(2n + 1) can be described by a Young tableau (the orthogonal group includes spinor representations). However, all the preceding formulas are valid in this case, too, if f_i is taken to mean the eigenvalue of the diagonal operator X^i_i for the highest-order vector of the irreducible representation.

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LASER WITH RADIATION DIAGRAM OF DIFFRACTION WIDTH

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As is well known [1,2], a large number of different modes whose resonant frequencies lie inside the luminescence line of the active medium, are excited simultaneously in a laser. This not only affects adversely the coherence of the radiation, but also distorts appreciably the directivity pattern, which becomes broad and jagged. The latter circumstance makes it difficult to use lasers for many scientific and technical applications.

The existing methods of selecting the oscillations for the purpose of inducing lasing conditions in one of the lower modes (TEM_{00q}) are based on the insertion of various optical elements into the resonator, and are incovenient in that they cause large losses. In this connection we consider a mode selection method based on choosing a resonator configuration such that the diffraction losses of the proper modes are essentially different. This property is possessed, in principle, by an ordinary confocal resonator, but only if its dimensions correspond to very small Fresnel numbers $N < 1$, where $N = r^2 L\lambda$, r is the radius of the mirrors, and L is the length of the resonator. For $r \sim 1$ cm, $\lambda = 10^{-4}$ cm, and $N \sim 1$ the resonator length L