# THEORY OF "STEPS" OF VOLTAGE-CURRENT CHARACTERISTIC OF THE JOSEPHSON TUNNEL CURRENT

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Dmitrenko, Yanson, and Svistunov [1], in a study of the Josephson tunnel effect [2], observed several singularities in the transition from superconducting tunneling to single-particle tunneling in a magnetic field. These singularities are manifest in the fact that the voltage-current characteristic of the tunnel junction consists of "steps," the distance between which along the voltage axis V is the same and does not depend on H, and the height of which along the current axis depends on the applied magnetic field, assuming a maximum value in a field proportional to the voltage of the step. In this paper we develop a theory of this phenomenon. We start from a hypothesis, advanced in [1] (see also [3]), whereby these effects are caused by the interaction between the Josephson alternating current and the field of the electromagnetic oscillations in a dielectric cavity between the superconductors.

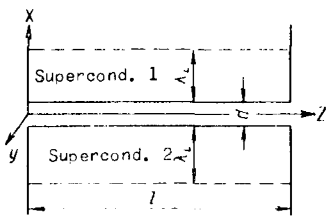


Fig. 1

The geometry of the tunnel junction is shown in Fig. 1. The magnetic field H is applied along the y axis, $H = H_y$. The electric field inside the junction can be regarded as constant and equal to $E_x = V/d$, where V is the applied bias. The Josephson tunneling is accompanied by a superconducting current whose magnitude, according to [2], is

$$j_x = j_s \sin\varphi \tag{1}$$

where $\varphi$ is the phase difference between superconductors 1 and 2. When $H \neq 0$ and $V \neq 0$ this current varies both in space and in time, causing in turn the appearance of an alternating electromagnetic field. The connection between the current and the field inside the junction is given by Maxwell's equation

$$-\frac{\partial H_y}{\partial z} = \frac{4\pi}{c} j_x + \frac{\epsilon}{c}\frac{\partial E_x}{\partial t} \tag{2}$$

According to Josephson [2]

$$V(z, t) = \frac{\hbar}{2e}\frac{\partial}{\partial t}; \quad H_y(z, t) = -\frac{\hbar c}{4e\lambda_L}\frac{\partial\varphi}{\partial z} \tag{3}$$

($\lambda_L$ is the London penetration depth, $\lambda_L \gg d$).

Substituting (3) in (2) and using (1), we obtain the equation

$$\frac{\partial^2\varphi}{\partial z^2} - \frac{1}{\bar{c}^2}\frac{\partial^2\varphi}{\partial t^2} = (\lambda_j)^{-2}\sin\varphi \tag{4}$$

where $\lambda_j^2 = \hbar c^2/16\pi e\lambda_L j_s$ is the square of the "Josephson depth of penetration" for weak superconductivity (see [2,4]), and $\bar{c} = c(d/2\epsilon\lambda_L)^{1/2}$ is the propagation velocity of the decelerated electromagnetic waves in the insulating layer between the superconductors (see [3,5]).

Equation (4), which is analogous to the corresponding equation of [3], generalizes the Ferrel-Prange equation [4] to include the non-stationary case. This equation can be used to describe the interaction between the Josephson current with the electromagnetic field it generates. Actually, Eq. (4) is nonlinear, so that the amplitude of the produced field (3) can be obtained in principle by solving this equation.

We investigate here the case when $\lambda_j$ is large compared with the width $l$ of the junction (see (10) below). The nonlinear term $\lambda_j^{-2}\sin\varphi$ in (4) can then be calculated by perturbation theory. We note, however, that a term describing the damping must be added in this case to the left side of (4); we write this term in the form $-(1/\bar{c}^2)(1/\tau)(\partial\varphi/\partial t)$, where $\tau$ is the characteristic relaxation time, proportional to the quality factor Q for the system ($Q = \omega\tau$). The solution of (4) can be obtained in the form

$$\varphi(z, t) = \varphi_0 - kz + \omega t + \Phi(z, t) \tag{5}$$

where k and $\omega$ are the wave vector and frequency of the Josephson current, and are proportional, according to (3), to the external magnetic field H and to the constant potential difference V; $\Phi(z, t)$ is a small increment. For the dc component of the Josephson current

$$\bar{j} = (1/l)\int_0^l \overline{j(z, t)}dz \cong j_s(1/l)\int_0^l \overline{\cos(\varphi_0 - kz + \omega t)\Phi(z, t)}dz$$

(the bar denotes averaging with respect to time) we obtain ($\omega \neq 0$):

$$\bar{j}(\omega, H) \cong j_s(\bar{c}^2/4\lambda_j^2)\sum_{n=1}^{\infty}(a_n^2 + b_n^2)\frac{\omega/\tau}{(\omega_n^2 - \omega^2)^2 + \omega^2/\tau^2} \tag{6}$$

where $\omega_n$ are discrete frequencies ($\omega_n = \bar{c}(\pi n/l)$) and $a_n$ and $b_n$ are the coefficients of the expansions of sin(kz) and cos(kz) in seriers in $\cos(\pi nz/l)$ in the interval $(0, l)$; their values are

$$a_n = \frac{1 - \cos(k + k_n)l}{(k + k_n)l} + \frac{1 - \cos(k - k_n)l}{(k - k_n)l}$$

$$b_n = \frac{\sin(k + k_n)l}{(k + k_n)l} + \frac{\sin(k - k_n)l}{(k - k_n)l}; \quad k_n = \frac{\pi n}{l} \tag{7}$$

The form of the dependence of $\bar{j}$ on $\omega = 2eV/\hbar$ is shown in Fig. 2. It consists of a series of resonant maxima at biases $V_n = \hbar\omega_n/2e$ (we note that experiments in which the current is specified and V is measured (see[1]) yield the dashed steps shown in Fig. 2). Near the n-th "resonance" ($\omega \approx \omega_n$) the field distribution takes the form $V'(z, t) \approx A_n(t)\cos(\pi nz/l)$. In this case V' has an antinode at the boundaries of the junction [1]. According to [1], such waves have the lowest losses, i.e., the largest Q, so that their intensity is large compared with the other modes of oscillations. As seen from (6) and (7), the value of the n-th maximum (step) at $\omega = \omega_n$ is

$$\bar{j}_n^{max}(H) = j_s(l/2\pi n\lambda_j)^2 Q_n \cdot F_n^2(\Phi/\Phi_0) \tag{8}$$

where $\Phi = 2H\lambda_L l$ is the magnetic flux inside the junction, $\Phi_0$ is the quantum of the magnetic

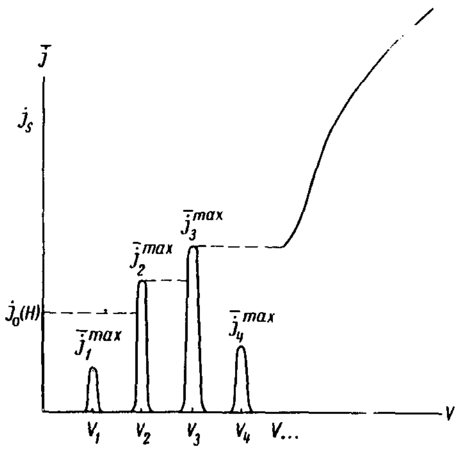


Fig. 2

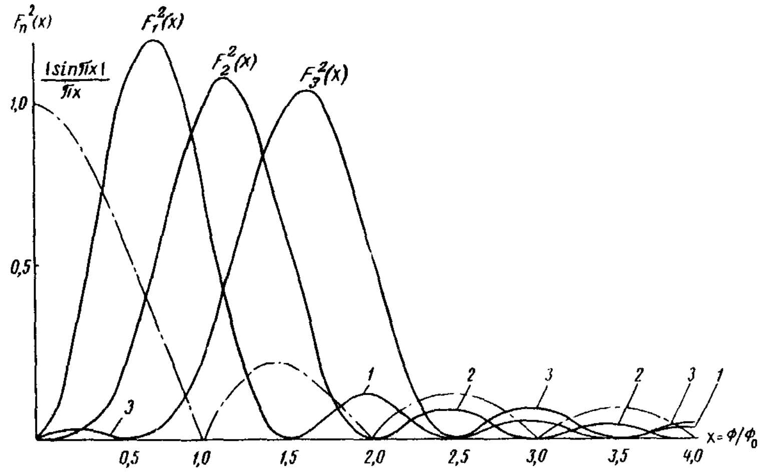


Fig. 3

flux (= $\hbar c/2e$), and $F_n(x)$ is given by

$$F_n(x) = (2/\pi)[x/|x^2 - (n/2)^2|] \cdot \begin{cases} |\cos \pi x|, & n = 1, 3, 5, \ldots \\ |\sin \pi x|, & n = 2, 4, 6, \ldots \end{cases} \tag{9}$$

The formula obtained is valid, as is clear from its derivation, if $\bar{j}_n(H)$ is small compared with $j_s$, i.e., if

$$(\ell/2\pi n\lambda_j)^2 Q_n \ll 1 \tag{10}$$

If the Q is not too large ($Q \lesssim 10$), then inequality (10) is satisfied even for relatively thick junctions, $\ell \sim \lambda_j$. According to (8), the dependence of the height of the steps on the magnetic field is determined by the square of the function $F_n(x)$, where $x = \Phi/\Phi_c = (2e\lambda_L \ell/\pi\hbar c)H$. The functions $F_n(x)$ are plotted in Fig. 3 (the dashed curves in the same figure show the dependence of the Josephson dc current $j_0$ at V = 0 on the magnetic field [6,8]). When n is large the function $F_n(x)$ has a principal maximum approximately equal to unity at $x \approx n/2$, and an infinite number of secondary maxima of much lower intensity (Fig. 3).

It is easy to see from (6) and (8) and from Figs. 2 and 3 that the results explain the following experimental facts observed in [1]: 1) The multiplicity of the voltage steps, $V = V_n = (\hbar/2e)\bar{c}(\pi n/\ell)$; 2) the maximum height of the n-th step in a magnetic field proportional to $V_n$, with $H = H_n \approx H_n(e/2\lambda_L d)^{1/2}$ (cf. also [3]); 3) the alternation of the minima and maxima of the heights of the steps as a function of the magnetic field. In addition, formula (8) determines the dependence of the heights of the steps on the magnetic field, a dependence which can be verified experimentally. It is now desirable to carry out the experiments under conditions in which the voltage V is specified and the current is measured, so as to observe a voltage-current characteristic of a "resonant" type shown in Fig. 2.

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## MONOTONICITY OF THE DECAY OF UNSTABLE PARTICLES CORRESPONDING TO AN n-ORDER POLE

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It was pointed out in a recent paper [1] that the energy distribution of unstable particles $\omega(E)$ may possibly correspond not to a first-order pole, as is usually assumed, but to a pole of order n:

$$\omega_n(E) = C_n[(E - E_0)^2 + (\Gamma^2/4)]^n \tag{1}$$

In order for these values of $\omega_n(E)$ actually to correspond to the energy distribution of the unstable particle, it is necessary, of course [2], that the decay of such a particle be monotonic 1), i.e.,

$$[dL_n(t)]/dt \leq 0, \qquad t \in [0, \infty) \tag{2}$$

where $L_n(t) = |p_n(t)|^2$ is the law of decay of the unstable particle with energy distribution (1). In [2] we presented a complete description of all the energy distributions $\omega(E)$ of the unstable particles, i.e., all the $\omega(E)$ for which the monotonicity condition is satisfied. We can prove, by verifying the necessary and sufficient conditions indicated in [2], that the $\omega_n(E)$ belong to the class of energy distributions of physical systems.

It is simpler, however, to prove this directly. It was shown in [1] that 2)

$$p_n(t) = \exp\left\{-\frac{i}{\hbar}E_0 t - \frac{\Gamma t}{\hbar}\right\}\sum_{\ell=0}^{n-1}\left(\frac{\Gamma t}{\hbar}\right)^{n-\ell-1}\frac{(n+\ell-1)!(n-1)}{(n-\ell-1)!\ell!(2n-2)!} \tag{3}$$

and consequently

$$\frac{dL_n(t)}{dt} = -\frac{\Gamma}{\hbar}\exp\left\{-\frac{\Gamma}{\hbar}\right\}\sum_{k=0}^{n-1}\left(\frac{\Gamma t}{\hbar}\right)^{n-k-1}\frac{(n+k-1)!(n-1)!}{(n-k-1)!k!(2n-2)!}$$
$$\times\left\{\sum_{\ell=0}^{n-1}\left(\frac{\Gamma t}{\hbar}\right)^{n-\ell-1}\frac{(n+\ell-1)!(n-1)!}{(n-\ell-1)!\ell!(2n-2)!} - 2\sum_{\ell=0}^{n-2}\left(\frac{\Gamma t}{\hbar}\right)^{n-\ell-2}\frac{(n-\ell-1)(n+\ell-1)!(n-1)!}{(n-\ell-1)!\ell!(2n-2)!}\right\} \tag{4}$$