[3] V. Bargmann and E. P. Wigner, Proc. Nat. Ac. Sci. 34, 211 (1948).
[4] M. A. B. Beg and A. Pais, Phys. Rev. Lett. 14, 509 (1965).
[5] Delbourgo, Rashid, and Strathdee, Phys. Lett. 14, 719 (1965); R. Blankenbeckler et al., Phys. Rev. Lett. 14, 518 (1965); R. Oehme, ibid 14, 664 (1965); Nguyen Van Hieu, JINR Preprint, 1965.
[6] I. I. de Swart, Revs. Mod. Phys. 35, 916 (1963).
[7] R. G. Goldstein and N. R. Lipshutz, Preprint, 1965.
[8] V. Gupta and V. Singh, Phys. Rev. 135, 1442 (1964).

SYMMETRY OF THE HYDROGEN ATOM

I. A. Malkin and V. I. Man'ko
Moscow Physico-technical Institute
Submitted 8 July 1965

In view of the success of the group approach to elementary-particle physics, questions connected with the symmetry of wave equations are presently attracting much attention.

Fock [1] and Bargmann [2] have shown that in a Coulomb potential the wave functions belonging to one level realize a finite-dimensional representation of a compact group O_4, which has been regarded as the symmetry group for this problem.

Barut et al. have shown in a recent paper [3] that the states of the discrete spectrum of the hydrogen atom form a basis of an infinite-dimensional representation of the deSitter algebra (4 + 1). As shown by Thomas [4], the basis of the infinite-dimensional representation of the deSitter group S is formed by matrix elements of representations of the compact group O_4 contained in it.

The purpose of the present paper is to show that the "symmetry group" of the hydrogen atom is the non-compact group O_6, the Lie algebra of which is the algebra D_3, and to present a simple construction showing that the functions belonging to the discrete spectrum form a single infinite-dimensional irreducible representation of this algebra.

Let $\varphi(x_1 \ldots x_n)$ satisfy the equation

$$\hat{A}\varphi = 0 \tag{1}$$

where $\hat{A}$ is a linear differential operator. We define as the symmetry group of Eq. (1) the aggregate of operators $\hat{M}_\alpha$ forming an algebra closed against commutation and satisfying the condition

$$[\hat{A}\hat{M}_\alpha]\varphi = 0 \tag{2}$$

If φ is a solution, then $\hat{M}_\alpha\varphi$ is also a solution.

As shown by Fock[1], the eigenfunctions of the discrete spectrum of the hydrogen atom, in the momentum representation and in the variables ζ_i (i = 1, ... 4)

$$\vec{\zeta} = [2p_0(p_0^2 + p^2)^{-1}\vec{p}, \qquad \zeta_4 = (p_0^2 - p^2)(p_0^2 + p^2)^{-1}$$

are homogeneous harmonic polynomials of the variables ζ_i of degree N - 1 (N is the principal quantum number), i.e., they are solutions of the equation

$$\Delta\varphi \equiv \frac{\partial^2\varphi}{\partial\zeta_1^2} + \frac{\partial^2\varphi}{\partial\zeta_2^2} + \frac{\partial^2\varphi}{\partial\zeta_3^2} + \frac{\partial^2\varphi}{\partial\zeta_4^2} = 0 \tag{3}$$

It is easy to check directly that the 15 operators

$$M_{ik} = -\,i(\zeta_i \frac{\partial}{\partial\zeta_k} - \frac{\partial}{\partial\zeta_i})$$

$$I_i = \zeta_k^2 \frac{\partial}{\partial\zeta_i} - 2\zeta_i\zeta_k \frac{\partial}{\partial\zeta_k} - 2\zeta_i \tag{4}$$

$$P_i = -i\frac{\partial}{\partial\zeta_i}, \quad I = \zeta_k \frac{\partial}{\partial\zeta_k} + 1$$

commute on the solutions of (3) with four-dimensional Laplacian $\Delta\varphi$.

The construction of the operators (4) was suggested to the authors by the analogy between Eq. (3) and the Klein-Gordon equation for a particle with zero mass [5,6].

We introduce the following operators

$$L_{ik} = -L_{ki} \quad (ik = 1,\ldots 6), \quad L_{ik} = M_{ik} \quad (i,k = 1,\ldots 4),$$
$$L_{i5} = \frac{1}{2}(I_i + iP_i), \quad L_{56} = -I, \quad L_{i6} = \frac{1}{2}(P_i + iI_i) \tag{5}$$

which satisfy the standard commutation relations of the algebra D_3

$$[L_{ik}L_{em}] = i(\delta_{ie}L_{km} + \delta_{km}L_{ie} - \delta_{im}L_{ke} - \delta_{ke}L_{im}) \tag{6}$$

The noncompact group written out above is the symmetry group of the hydrogen atom in the sense of (1) and (2). The hydrogen-atom eigenfunctions corresponding to the principal quantum number N = n + 1 can be realized as irreducible tensor degrees $\Pi^n_{i_1i_2\ldots i_n}$ of the vector ζ_i ($\zeta_i = 1$), with $\Pi^n_{i_1i_2\ldots i_n}$ a fully symmetrical tensor and $\delta_{i_1i_2}\cdot\Pi^n_{i_1i_2\ldots i_n} = 0$. We present a summary of the matrix elements of the operators (4) in this basis:

$$I_i\,\Pi^n_{i_1\ldots i_n} = \delta_{ii_{n+1}}\,\Pi^{n+1}_{i_1\ldots i_n i_{n+1}} \tag{7a}$$

$$P_i\,\Pi^n_{i_1\ldots i_n} = i\left[2n\sum_{k=1}^{n}\delta_{ii_k}\Pi^{n-1}_{i_1\ldots i_{k-1}i_{k+1}\ldots i_n} - 2\sum_{(j,k)}\delta_{i_ji_k}\Pi^{n-1}_{i_1\ldots i_{k-1}i_{k+1}\ldots i_{j-1}i_{j+1}\ldots i_n}\right] \tag{7b}$$

$$M_{ij}\,\Pi^n_{i_1\ldots i_n} = \sum_{k=1}^{n}(\delta_{ii_k}\,\Pi^n_{i_1\ldots i_{k-1}i_{k+1}\ldots i_n j} - \delta_{ji_k}\,\Pi^n_{i_1\ldots i_{k-1}i_{k+1}\ldots i_n i}) \tag{7c}$$

$$I\,\Pi^n_{i_1\ldots i_n} = (n+1)\Pi^n_{i_1\ldots i_n} \tag{7d}$$

In (7b), (j,k) denotes summation over combinations of the n indices taken two at a time.

The tensors $\Pi^n_{i_1 i_2 \ldots i_n}$ are normalized by the condition (7a) and $\Pi^0 = 1$. The operators I_i and P_i transform the level N into N + 1 and N - 1, respectively; this means that from any state we obtain in succession all the entire aggregate of states. That is to say, we have constructed an infinite-dimensional representation of the algebra of the operators (5) in the space $H = \epsilon\ \Pi^n$, where H is the direct sum of the spaces of the functions corresponding to the given space. This representation is irreducible, for if an invariant subspace were to exist, it would contain at least one tensor degree Π^n, action of which by the operators I_i and P_i would yield all the tensor degrees, i.e., the non-zero invariant subspace would coincide with all the space H. We present the values of the Casimir operators for this representation.

$$C_2 = L_{ik}L_{ki} \equiv 6, \quad C_3 = \epsilon_{ikemnp}L_{ik}L_{em}L_{pn} = 0, \quad C_4 = L_{ik}L_{km}L_{mn}L_{ni} = -12$$

The deSitter algebra S is a subalgebra of D_3, namely $S_{ij} = L_{ij}$ (i, j = 1,...5). It is remarkable that the representation constructed above remains irreducible also with respect to a subalgebra of S, as can be verified by letting the operators L_{i5} operate on 1. In addition, the Casimir operator for the deSitter algebra is

$$\hat{Q} = S_{ij}S_{ij} = 4 - \frac{1}{2}[(\zeta_i^2 + 1)^2 \frac{\partial^2}{\partial\zeta_k^2}$$

We see therefore that

$$\hat{Q}\cdot\Pi^n_{i_1\ldots i_n} = 4\Pi^n_{i_1\ldots i_n}$$

The second Casimir operator of the algebra S is equal to zero. This proves completely that the representation remains irreducible when we narrow down from D_3 to the deSitter algebra. We note that the algebra D_3 contains a subalgebra with the commutation relations of the algebra A_2, and therefore the levels of the hydrogen atom can also be classified with the aid of irreducible representations of this algebra.

The authors are deeply grateful to A. M. Baldin, V. B. Berestetskii, A. A. Komar, A. M. Perelomov, V. S. Popov, and I. S. Shapiro for a discussion of the results of the work, and also to M. A. Naimark for useful advice and consultations.

[1] V. Fock, Z. Physik 98, 145 (1935).
[2] V. Bargmann, Z. Physik 99, 576 (1936).
[3] Barut, Budini, and Fronsdal, Preprint TRIEST, Ic/65/34.
[4] L. H. Thomas, Ann. of Math. 42, 113 (1960).
[5] P. A. M. Dirac, Ann. of Math. 37, 429 (1936).
[6] V. I. Ogievetskii and I. V. Polubarinov, JETP 37, 470 (1959), Soviet Phys. JETP 10, 335 (1960).