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MOTION OF VORTEX FILAMENTS IN SUPERCONDUCTING ALLOYS

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It is difficult to apply the ordinary hydrodynamic approach [1] to the motion of vortex filaments in superconductors of type II, owing to the presence of the lattice. In the state of local equilibrium, the condensate is characterized by a superconducting velocity $\vec{v}_s$ and an electron density N. However, in view of the Coulomb interaction, the relaxation rate of N is so large that the only "hydrodynamic" quantity is $\vec{v}_s$. It is easy to obtain for $\vec{v}_s$ the equation

$$\dot{\vec{v}}_s = e/m\,(\vec{E} - \nabla\varphi),$$

but this is a trivial consequence of Maxwell's equations, since by definition

$$\mathrm{curl}\,\vec{v}_s + (e/m)\vec{H} = 0$$

($\vec{v}_s = 1/2\,m(\nabla x - 2e\vec{A})$, x is the phase of the ordering parameter, $\Delta = |\Delta|\exp(ix)$, $\vec{H} = \mathrm{curl}\,\vec{A}$, and $\hbar = c = 1$).

In view of the field character of the vortex, being a line of singularities of $v_s \sim 1/2mr$ [2], for the solution of the problem it is in fact sufficient to find directly the response of the superconducting system to an external field, i.e., the current $\vec{j} = \vec{j}\{\vec{A}\}$ as a functional of $\vec{A}$. In the case of "dirty" alloys ($\tau T_c \ll 1$), for physically reasonable frequencies $\omega\tau \ll 1$, this can be done by directly expanding $\vec{j}$ in terms of the small parameter ωt:

$$\vec{j} = \vec{j}_s\{\vec{v}_s\} + \vec{j}_n, \quad \vec{j}_n = \hat{\sigma}\{\vec{v}_s\}\vec{E},$$

where $\vec{j}_s\{\vec{v}_s\}$ is the equilibrium superconducting current in the specified magnetic field $\vec{A}$. The phase x in $\vec{v}_s$ is determined by the additional equation div $\vec{j}_s = 0$, which follows from the imaginary part of the equilibrium equation for the ordering parameter. The kernel of the conductivity operator $\hat{\sigma}\{\vec{v}_s\}$ can be shown to have the form

$$(-i)d/d\omega\,\langle j_i, j_k\rangle|_{\omega \to \Omega},$$

where the correlator $\langle j_i, j_k\rangle$ can also be calculated from the equilibrium state in the specified external field [1)]. The relations presented constitute, together with Maxwell's equations and the continuity equation, a complete system that yields the vortex equation of motion.

To derive these equations, it is convenient to start from the general principles of nonequilibrium thermodynamics (see, for example, [3]). The work performed on the system by the external current $\vec{j}_{ext}$ per unit time

$$\dot{W} = -\int dV\, \vec{j}_{ext}\cdot\vec{E},$$

can be expressed in terms of the response of the system

$$\dot{W} = (\partial/\partial t) \int dV \frac{H^2}{8\pi} + \int dV\, [\vec{j}_s\cdot\vec{E} + \vec{j}_n\cdot\vec{E}],$$

from which it follows, by virtue of $\vec{E} = (m/e)\dot{\vec{v}}_s + \nabla\varphi$ and div $\vec{j}_s\{\vec{v}_s\} = 0$, that

$$\dot{W} = (\partial/\partial t) \int dV[\frac{H^2}{8\pi} + F\{\vec{v}_s\}] + \int dV\, \hat{\sigma}\{\vec{v}_s\}E^2 \left(\frac{\delta}{\delta\vec{v}_s}\int dV\, F - \frac{m}{e}\vec{j}_s\right) \tag{1}$$

According to this equation, the maximum work of the reversible transition into the nonequilibrium state is equal to

$$W_{min} = \int dV\, [\frac{H^2}{8\pi} + F\{\vec{v}_s\}] = -\frac{\Delta S}{T} \tag{2}$$

(ΔS is the deviation of the entropy from the equilibrium value), whereas the loss due to the irreversible processes is represented in the form $T\Delta\dot{S} \simeq \int dV\, \hat{\sigma}\{\vec{v}_s\}E^2$. Calculation of the latter integral for a moving vortex yields

$$T\Delta\dot{S} = \int dl\, \eta v_L^2, \tag{3}$$

where the integration is along a vortex line and $\vec{v}_L$ is the velocity of the filament element.

Let us express the quantity W_{min} (2) in terms of the parameters of a filament situated in an external field. For extremely hard superconductors ($\kappa \gg 1$) the field of the filament is small ($\sim\lambda/\kappa$ in dimensionless variables [2]) and we can expand (2) in powers of this small quantity. Retaining only the terms containing the filament to second order of smallness, we obtain

$$W_{min} = \int dV\, [\frac{\vec{H}\delta\vec{H}}{4\pi} + \frac{m}{e}\vec{j}_s\delta\vec{v}_s] + \int dV\, [\frac{(\delta\vec{H})^2}{8\pi} + \frac{m}{2e}\delta\vec{j}_s\delta\vec{v}_s],$$

where $\delta\vec{H}$, $\delta\vec{j}_s$, and $\delta\vec{v}_s$ are quantities describing the vortex. Integration of this equation by parts with the aid of the formula $\delta\vec{H} = (-m/e)$ curl $(\delta\vec{v}_s)$ and Maxwell's equations leads, owing to the presence of the singularity in $\delta\vec{v}_s$, to the following result:

$$W_{min} = \frac{1}{4e}\int d\vec{l}\cdot\vec{H} + \epsilon_0\int dl \tag{4}$$

($\epsilon_0 = \delta H_m/8e$ is the self energy of the vortex per unit length and δH_m is the field at the center of the vortex). The second term obviously represents the "elastic" energy of the filament.

From the equations for the relaxation of the parameters of the partial equilibrium of the system (i.e., the coordinates $\vec{r}(\varphi, t)$ of the filament)

$$\dot{x}_i \equiv v_{Li} = \int dl'\gamma_{ik}\frac{\delta S}{\delta x'_k}$$

we get from (2), (3), and the Onsager principle of the symmetry of γ_{ik} [2]:

$$\vec{v}_L = \frac{T}{\eta}\frac{\delta S}{\delta r} = -\frac{1}{\eta}\frac{\delta W_{min}}{\delta r}$$

Therefore, varying (4) with respect to the filament element, we obtain ultimately

$$\eta\vec{v}_L = [\vec{j}_T \times \vec{\Phi}_0] + \epsilon_0 \frac{\vec{n}}{R} \tag{5}$$

where $\vec{j}_T \equiv \vec{j}_s$ is the external current, $\Phi_0 = \pi/e$ the magnetic flux of the vortex, $\vec{n}$ the principal normal to the vortex line, and R the radius of curvature of the filament. We note that in accord with the obtained expression, the effective mass of the filament naturally vanishes, in view of the approximation $\omega\tau \ll 1$. Owing to the nonlinearity of the functionals $\vec{j}_s\{\vec{v}_s\}$ and $\hat{\sigma}\{\hat{v}_s\}$, the coefficients η and ϵ_0 are generally speaking dependent on the external current. Simple estimates show that the limits of applicability of (5) are given by the inequalities $j_T \ll j_{cr}$ and $R \gg \lambda/\kappa$ (λ = depth of penetration of the field).

Upon application of a constant external current $\vec{j}_T$ on the mixed state in the superconductor, the average electric field intensity produced by the moving vortices is equal to $\vec{E}_{av} = -\vec{v}_L \times \vec{B}$, from which it follows, taking (5) into account, that $\vec{E}_{av} = \rho\vec{j}_T$. For the resistivity we have

$$\rho = \frac{\Phi_0 B}{\eta} = \frac{\pi B}{e\eta}$$

and near T_c, for fields that are not too strong, $\rho = \rho_n B/H_{c2}$ [1].

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1) In the local approximation these calculations were carried out by the author under the direction of J. Bardeen. In particular, the value obtained for the viscosity coefficient of the moving vortex near T_c was $\eta = \pi\sigma_n H_{c2}/e$, where σ_n is the conductivity of the normal metal.

2) The only antisymmetrical expression that need be considered, $\gamma_{ik} \sim e_{ikl}v_{Ll}$, leads to a force acting along the filament. The remaining expressions do not agree with the energy conservation law [1].