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1. One of the central questions in the theory of nonrenormalizable interactions (NRI) is whether the difficulties of this theory result from the inability to solve the corresponding dynamic equations outside the framework of perturbation theory, or whether these difficulties demonstrate that the equations themselves are unsuitable. The first point of view is the starting point of the now developing "peratization" method [1,2]; the real accomplishments of this approach are as yet far from obvious.

The second possibility is that it is impossible to use in the NRI theory the concepts and quantities on which the dynamic description is based, for example the scattering matrix for the final time interval (see [3], where this point of view is formulated as applied to arbitrary point interactions). Accordingly, the dynamic field equations should be replaced by more general - axiomatic - equations, the derivation of which does not call for the use of the directly unobservable quantities referred to above.

We investigate this question here with the aid of the previously developed [4] differential (with respect to charge) axiomatic method using the simplest NRI model - scattering of two nonrelativistic particles. The matrix element of the renormalized interaction Lagrangian in the in-representation,  $L_{in}(x)$ , is chosen in the form

$$\langle \vec{k}, -\vec{k} | L_{in}(0) | \vec{k}', -\vec{k}' \rangle = -g(\vec{k} \cdot \vec{k}'). \quad (1)$$

Here  $g$  is the physical charge, and the particle mass is set equal to unity.

2. Landau [3] obtained the following equation for the scattering phase shift  $\delta(k)$  <sup>1)</sup>:

$$\delta''(k)/\delta'(k) = \frac{4}{\pi} k^2 \int_0^\infty \frac{dp \delta'(p)}{p(p^2 - k^2)} \quad (2)$$

with initial condition that follows from (1):

$$\delta'(k) \Big|_{g=0} = -k^3. \quad (3)$$

Differentiating (2) with respect to  $g$ , we arrive at

$$(\delta''/\delta')' - \frac{1}{2}(\delta''/\delta')^2 + 2[(\delta')^2 - C^2 k^2] = 0, \quad (4)$$

where  $C$  is the limit of  $\delta'(p)/p$  as  $p \rightarrow \infty$ . The term  $2C^2 k^2$  is the result of the integral

$$I = \frac{8}{\pi} k^2 \left\{ \int_0^\infty dp \int_0^\infty dq - \int_0^\infty dq \int_0^\infty dp \right\} \frac{p \delta'(p) \delta'(q)}{q(p^2 - k^2)(q^2 - p^2)}.$$

This quantity differs from zero in the case of non-uniform convergence of the integrals,

which prevents the interchange of the order of integration.

It is essential that when  $C \neq 0$  the scattering matrix  $S(t, -\infty)$  with finite  $t$  is non-unitary: according to [4], the quantity

$$\frac{d}{dg}[S^+(t, -\infty) S(t, -\infty)]$$

is expressed in terms of a difference of integrals, similar to I, and is proportional to  $C^2$ . Therefore the dynamic solution that can be obtained from the Schrodinger equation must correspond to  $C = 0$ .

In the latter case Eq. (4) yields  $\delta'(k) = A/[1 + (Ag + B)^2]$ , where A and B are functions of the momentum, determined by the initial conditions when  $g = 0$ . As a result we obtain the well known expression for the scattering amplitude

$$f(k) = \frac{g\delta_0'(k)}{k} \left[ 1 - g \frac{2}{\pi} k^2 \int_0^\infty \frac{dp \delta_0'(p)}{p(p^2 - k^2 - i\epsilon)} \right]^{-1}, \quad (5)$$

where  $\delta_0'(k) = \delta'(k)|_{g=0}$ . Substitution of (3) in this expression leads to a meaningless divergent result. By regularization (corresponding to separating in (1) an additional factor  $v^*(k')v(k)$  with a function  $v(k)$  that decreases sufficiently rapidly when  $k > L$ ) we arrive at the expression

$$f(k) = gk^2 / (1 - gik - g \frac{2k^2}{\pi} L),$$

which vanishes in the limit as  $L \rightarrow \infty$ .

3. We shall now seek solutions of (4) which cannot be obtained from the Schrodinger equation ( $C \neq 0$ ). A solution of this kind actually exists. It corresponds to  $\delta'(k) = -k^3/(1 + \xi^2)$  and is of the form ( $\xi = (3g)^{\frac{1}{3}}k$ )

$$f(k) = \frac{1}{2ik} \left( \frac{1 + i\xi}{1 - i\xi} \exp(-2i\xi) - 1 \right). \quad (6)$$

This solution, as expected, turns out to be non-analytic in the charge at the point  $g = 0$ . It vanishes following a regularization that leads to  $C = 0$ . This solution is obtained only when  $g > 0$ , corresponding to the initial assumption made in the derivation of (2), that there are no bound states.

4. A qualitatively similar situation arises in the relativistic problem of scattering with four-fermion interaction, if we confine ourselves to allowance for the two-particle intermediate states only. Equations (2) and (4) remain of the same form if we replace  $k^2$  by  $E - 2m$  ( $E$  = total c.m.s. energy,  $m$  = particle mass). Corresponding to a solution similar to (6) are the following limiting expressions for the phase:

$$\delta(E) \sim \begin{cases} \xi^2 (1 - \frac{4}{3}\xi + 2\xi^2 \ln \xi + \dots) & (\xi \ll 1) \\ \frac{1}{\xi^2} (1 + \frac{15}{512} \frac{1}{\xi} + \dots) & (\xi \gg 1) \end{cases} \quad (7)$$

Here  $\xi = E\sqrt{g}$ , and the limit as  $m \rightarrow 0$  is considered. A similar investigation of this problem, with an estimate of the contribution of many-particle intermediate states, is now under way.

5. It follows from the results that, at least within the framework of the models considered, the NRI problem has a singular solution free of the difficulties inherent in these interactions and non-analytic in the charge. This solution cannot be obtained from the dynamic equations and arises only when the problem is axiomatically formulated. The results confirm the likelihood of this point of view on the difficulties of the NRI theory, which starts from the assumption that the dynamic theory is not suitable for a description of such interactions.

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1) The index  $l$  has been omitted from the symbol for the phase; in the model under consideration, only the p-wave is significant. The prime denotes differentiation with respect to  $g$ .

### Erratum

In the article by D. A. Kirzhnits and M. A. Livshitz, Vol. 4, No. 2, p. 46, line 24, should read: "Kirzhnits [4] obtained..." in lieu of "Landau [3] obtained..."