DRIFT STATIONARY SOLUTIONS IN THE THEORY OF WEAK TURBULENCE

A.B. Kats and V.M. Kontorovich
Institute of Radiophysics and Electronics, Ukrainian Academy of Sciences
Submitted 3 August 1971
ZhETF Pis. Red. 14, No. 6, 392 - 395 (20 September 1971)

In the theory of weak turbulence [1, 2], as shown by Zakharov [3], the collision integral I for waves in the case of isotopic N_ω distributions has symmetry in frequency space, making it possible to factor out this integral, and in turn to find stationary non-equilibrium distributions $N_\omega = \omega^s$ (we omit the proportionality coefficients) for surface gravitational [4] and capillary [5] waves, turbulent plasma [6], and "acoustic" turbulence in a compressible liquid [7]. These distributions are similar to the Kolmogorov distribution [8, 9] in the hydrodynamic local-isotropic turbulence, and correspond to a constant energy flux [4 - 6] and particle flux [10, 11] in the inertial part of the spectrum.

The equilibrium distribution $N^0 = \omega^{-1}$ corresponds also to a more general distribution $N = [(\omega - \vec{k}\cdot\vec{u} - \mu)/(1 + \theta)]^{-1}_{\to}$, which causes the collision integral to vanish at arbitrary drift parameters $\vec{u}$, μ, and θ, corresponding to equilibrium at nonzero total momentum in the system with changed temperature and particle number.

For non-equilibrium distributions ($s \neq -1$), however, the corresponding statement does not hold for u and $\mu \neq 0$. It is shown below that solutions of the drift type (in the approximation linear in $\vec{u}$ and μ)

$$N(\mathbf{k}) = \omega^s(1 + \mu\omega^t + \vec{\kappa}\,\mathbf{u}\,\omega^p), \quad \kappa = \mathbf{k}/k, \tag{1}$$

which cause the collision integral to vanish, nevertheless exist. The exponents t and p are obtained by factorizing the collision integral I, making use of its symmetry in k-space on the classical distributions (1). It turns out that generally speaking, (1) does not correspond in any way to expansion of the distribution N_ω when ω is replaced by $\omega - \vec{k}\cdot\vec{u} - \mu$. Furthermore, besides the drift deviations, analogous deviations from isotropic equilibrium distribution also arise, but these will not be discussed here.

In the case of four-particle interaction (for a non-decaying spectrum) [1, 2, 4, 6]

$$I = \int d\tau_k \delta_k U_{\mathbf{k}\mathbf{k}_1|\mathbf{k}_2\mathbf{k}_3}[N_1N_2N_3 + NN_2N_3 - NN_1N_2 - NN_1N_3], \tag{2}$$

where $\delta_k \equiv \delta(\vec{k} + \vec{k}_1 - \vec{k}_2 - \vec{k}_3)\delta(\omega + \omega_1 - \omega_2 - \omega_3)$, $N \equiv N(\vec{k})$, $N_1 \equiv N(\vec{k}_1)$, etc.,

$$U_{\mathbf{k}\mathbf{k}_1\,\mathbf{k}_2\mathbf{k}_3} = U_{\mathbf{k}_1\mathbf{k}|\mathbf{k}_2\mathbf{k}_3} = U_{\mathbf{k}\mathbf{k}_1|\mathbf{k}_3\mathbf{k}_2} = U_{\mathbf{k}_2\mathbf{k}_3\mathbf{k}\mathbf{k}_1}. \tag{3}$$

For the distribution (1), the linearized integral (2) takes the form

$$I = I_0 + \mu I_\mu + \mathbf{u}\mathbf{I}, \tag{4}$$

where

$$I_\mu = \int d\tau_k \delta_k U_{\mathbf{k}\mathbf{k}_1|\mathbf{k}_2\mathbf{k}_3}(\xi\omega^t + \xi_1\omega_1^t - \xi_2\omega_2^t - \xi_3\omega_3^t), \tag{5}$$

$$\xi \equiv \xi(\omega, \omega_1, \omega_2, \omega_3) = \omega^s[(\omega_2\omega_3)^s - (\omega_1\omega_2)^s - (\omega_1\omega_3)^s],$$

$$d\tau_k = d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3.$$

$$\xi_1 \equiv \xi(\omega_1, \omega, \omega_2, \omega_3), \quad \xi_2 \equiv \xi(\omega_2, \omega_3, \omega, \omega_1),$$
$$\xi_3 = \xi(\omega_3, \omega_2, \omega, \omega_1),$$

I_0 is obtained, apart from the coefficient 3, from I_μ by replacing t with zero,

$$\mathbf{I} = \int dr_k \delta_k U_{kk_1|k_2k_3}(\vec{\kappa}\xi\omega^p + \vec{\kappa}_1\xi_1\omega_1^p - \vec{\kappa}_2\xi_2\omega_2^p - \vec{\kappa}_3\xi_3\omega_3^p). \tag{6}$$

Owing to the homogeneity of the dispersion law $k = \omega^\alpha$ and the transition probability

$$U_{\gamma k \gamma k_1|\gamma k_2 \gamma k_3} = \gamma^m U_{kk_1|k_2k_3},$$

the integrals (4) can be factored and represented in the form

$$I_\mu = \frac{1}{4}\omega^{\nu t}\int dr_k \delta_k U_{kk_1|k_2k_3}(\xi\omega^t + \xi_1\omega_1^t - \xi_2\omega_2^t - \xi_3\omega_3^t) \times \times(\omega^{-\nu_t} + \omega_1^{-\nu_t} - \omega_2^{-\nu_t} - \omega_3^{-\nu_t}), \tag{7}$$

$$I_u = \frac{1}{4}\omega^{\nu_p}\int dr_k \delta_k U_{kk_1|k_2k_3}(\vec{\kappa}\xi\omega^p + \vec{\kappa}_1\xi_1\omega_1^p - \vec{\kappa}_2\xi_2\omega_2^p - \vec{\kappa}_3\xi_3\omega_3^p) \times \times(\vec{\kappa}\omega^{-\nu_p} + \vec{\kappa}_1\omega_1^{-\nu_p} - \vec{\kappa}_2\omega_2^{-\nu_p} - \vec{\kappa}_3\omega_3^{-\nu_p}). \tag{8}$$

We use here the notation

$$\mathbf{I} = \vec{\kappa} I_u, \quad \nu_t = \nu + t, \quad \nu_p = \nu + p, \quad \nu = am + ad(n-1) - 1 + (n-1)s, \tag{9}$$

in which d is the dimensionality of the vector $\vec{k}$, and n is the number of particles taking part in the process, equal to four in (7) and (8). The transformation of (5) and (6) into (7) and (8) consists of rotations and stretchings, which consecutively transform $\vec{k}_i$ into $\vec{k}$ with conservation of similarity of the polygon expressing the law of momentum conservation. Use is made here both of the symmetry (3) of the transition probability, and of the symmetry relation between the incoming and outgoing terms in (2). For I_0 this transformation causes the integral to take the form

$$I_0 = \frac{\omega^\nu}{4}\int dr_k \delta_k U_{kk_1 k_2k_3}(\omega\omega_1\omega_2\omega_3)^s(\omega^{-s} + \omega_1^{-s} - \omega_2^{-s} - \omega_3^{-s}) \times \times(\omega^{-\nu} + \omega_1^{-\nu} - \omega_2^{-\nu} - \omega_3^{-\nu}), \tag{10}$$

obtained in [3, 4] with the aid of the Zakharov transformation in ω space. It is seen from (7) and (6) that there exist solutions with $\nu_t = 0$ and -1, and $\nu_p = -\alpha$.

An analogous transformation for three-particle processes (in the case of a decay spectrum) leads to the expression

$$I_u^{(3)} = \frac{\omega^{\nu_p}}{3}\int dr_k^{(3)}\delta_k^{(3)}U_{k|k_1k_2}(\vec{\kappa}\xi^{(3)}\omega^p - \vec{\kappa}_1\xi_1^{(3)}\omega_1^p - \vec{\kappa}_2\xi_2^{(3)}\omega_2^p) \times \times(\vec{\kappa}\omega^{-\nu_p} - \vec{\kappa}_1\omega_1^{-\nu_p} - \vec{\kappa}_2\omega_2^{-\nu_p}), \tag{11}$$

where

$$d\tau_k^{(3)} = dk_1 dk_2, \quad \delta_k^{(3)} = \delta(k - k_1 - k_2)\,\delta(\omega - \omega_1 - \omega_2),$$

$$\xi^{(3)} = -\omega^s(\omega_1^s + \omega_2^s), \quad \xi_1^{(3)} = \omega_1^s(\omega^s - \omega_2^s), \quad \xi_2^{(3)} = \omega_2^s(\omega^s - \omega_1^s),$$

and ν and ν_p are given by formula (9) with n = 3. It follows from (11) that a solution with $\nu_p = -\alpha$ exists.

Denoting the exponents s, p, and t corresponding to $\nu = 0$ by s_0, p_0, and t_0, and those corresponding to $\nu = -1$ by s_1, p_1, and t_1, we verify that $p_0 = -\alpha$, $t_0 = -1$, $p_1 = 1 - \alpha$, and $t_1 = 1$.

The drift solutions corresponding to known stationary distributions are listed in the table.

	Gravitational waves	Capillary waves	Acoustic turbulence	Plasma oscillations
d	2	2	3	3
α	2	2/3	1	1/2
n	4	3	3	4
m	6	9/2	3	4
s_0	-23/3 [10]	-	-	-11/6 [11]
p_0	-2	-	-	-1/2 1)
t_0	-1	-	-	-1 1)
s_1	-8 [4]	-17/6 [5]	-9/2 [7]	-13/6 [6]
p_1	-1	1/3	0	1/2
t_1	1	1	1	1

1) It is curious to note that in this case the solution coincides formally with the "equilibrium" solution (see [11]).

The solutions obtained make it possible to consider the problem of stability of stationary distributions and of second sound in a weakly turbulent system of waves. Oscillations of surface waves of the second-sound type will be treated in a separate communication by V.K. Gavrikov, Yu.A. Sinitsyn, and the authors.

[1] A.A. Vedenov, Voprosy teorii plazmy (Problems of Plasma Theory) 2, 203, Atomizdat, 1963.
[2] B.B. Kadomtsev, ibid. 4, 188, Atomizdat, 1964.
[3] V.E. Zakharov, PMTF No. 4, 35 (1965).
[4] V.E. Zakharov and N.N. Filonenko, Dokl. Akad. Nauk SSSR 170, 1292 (1966) [Sov. Phys.-Dokl. 11, 881 (1967)].
[5] V.E. Zakharov and N.N. Filonenko, PMTF No. 5, 62 (1967).
[6] V.E. Zakharov, Zh. Eksp. Teor. Fiz. 51, 688 (1966) [Sov. Phys.-JETP 24, 455 (1967)].
[7] V.E. Zakharov and R.Z. Sagdeev, Dokl. Akad. Nauk SSSR 192, 297 (1970) [Sov. Phys.-Dokl. 15, 439 (1970)].
[8] A.N. Kolmogorov, ibid. 30, 299 (1941).
[9] I.M. Yaglom and A.S. Monin, Statisticheskaya gidromekhanika (Statistical Hydromechanics), Nauka, 1965.
[10] V.E. Zakharov, Dissertation, Novosibisrk, 1967.
[11] E.A. Kaner and V.M. Yakovenko, Zh. Eksp. Teor. Fiz. 58, 582 (1970) [Sov. Phys.-JETP 31, 313 (1970)].