Analytical Model of the Time Developing Turbulent Boundary Layer

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We present an analytical model for the time-developing turbulent boundary layer (TD-TBL) over a flat plate. The model provides explicit formulae for the temporal behavior of the wall-shear stress and both the temporal and spatial distributions of the mean streamwise velocity, the turbulence kinetic energy and Reynolds shear stress. The resulting profiles are in good agreement with the DNS results of spatially-developing turbulent boundary layers at momentum thickness Reynolds number equal to 1430 and 2900 [5 – 7]. Our analytical model is, to the best of our knowledge, the *first* of its kind for TD-TBL.

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Formulation of the problem. In this paper we derive an analytical model for the time-developing turbulent boundary layer (TD-TBL) over a flat plate. We consider a flat plate ($z = 0$) submerged in an incompressible viscous fluid at rest for time $t < 0$. At time $t = 0$ the fluid moves as a whole in the x-direction with velocity V_∞. This motion creates a boundary layer near the plate. We assume that this boundary layer is turbulent. Note that in the case of TD-TBL all statistical characteristics of the flow depend only on the time t and distance z from the wall [e.g. the mean streamwise velocity is $V(z,t)$]. In contrast, in a spatially-developing TBL (SD-TBL) the statistical characteristics depend on two spacial directions, z and x, and the mean velocity has also V_z component normal to the wall. Nevertheless in the limit of large Reynolds number both the TD-TBL and SD-TBL become asymptotically equivalent (with the replacement $x \leftrightarrow V_\infty t$) [1]. Therefore it is reasonable to first consider the simpler simple case of TD-TBL.

Definitions and Model Equations. We start from the Navier-Stokes equations for an incompressible fluid. The velocity $\mathbf{U}(\mathbf{r},t)$ is decomposed into the sum of its mean value $\mathbf{V}(z,t) \equiv \langle \mathbf{U}(\mathbf{r},t)\rangle$ and a fluctuating part $\mathbf{u}(\mathbf{r},t)$, $\mathbf{U}(\mathbf{r},t) = \widehat{\mathbf{x}}V(z,t) + \mathbf{u}(\mathbf{r},t)$ Here $\widehat{\mathbf{x}}$ is the unit vector in x-direction, $\mathbf{r} = \{x\,, y\,, z\}$ is three dimensional coordinate, and $\langle \dots \rangle$ denotes averaging in time and in the span-wise direction y.

The three main quantities in the model are the mean shear $S(z,t)$, the tangential Reynolds stress $\tau(z,t)$ and the turbulence kinetic energy per unit mass $K(z,t)$, defined as:

$$S(z,t) \equiv \frac{\partial V}{\partial z},\ \tau(z,t) \equiv -\langle u_x u_z\rangle,\ K(z,t) = \frac{1}{2}\,\langle |\mathbf{u}|^2\rangle. \tag{1}$$

The mean momentum equation (e.g. Eq. (4.12) in [2]) after integration over z has the form:

$$\nu S(z,t) + \tau(z,t) = \tau_*(t) + \frac{\partial}{\partial t}\int_0^z V(z',t)\,dz'\ . \tag{2}$$

Here ν is the kinematic viscosity and the right hand side (RHS) is momentum flux toward the wall. The integration constant $\tau_*(t) = \nu S(0,t)$ is the wall shear stress.

The turbulence kinetic energy conservation equation for a TD-TBL (see e.g. Eq. (5.132) in [2]) can be written as:

$$\partial K(z,t)/\partial t + \mathcal{E}(z,t) + \boldsymbol{\nabla}\cdot \mathbf{T}(z,t) = \tau(z,t)\,S(z,t). \tag{3}$$

The RHS of this equation represents the production of turbulence kinetic energy by the mean shear. The two terms in the left hand side (LHS): the rate of energy dissipation, $\mathcal{E}$, and spatial energy flux, $\mathbf{T}$, require modeling via S, τ and K.

Equations (2) and (3) for S and K are exact. In order to solve them, we need to add a third equation for $\tau(z,t)$ and model both $\mathcal{E}(z,t)$ and $\mathbf{T}(z,t)$. It is reasonable to assume that in the log-layer (region where the von-Kármán log-law holds) the shear and normal Reynolds stresses have the same z and t dependence, thus their ratio is constant: $\tau(z,t)/K(z,t) = c^2$.

As suggested by [2] we write the rate of turbulence kinetic energy dissipation as $\mathcal{E}(z,t) = b\,[K(z,t)]^{3/2}/z$,

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where b is a positive constant and the spatial energy flux in the z direction as $T(z,t) = -D(z,t)\,\partial K(z,t)/\partial z$, where $D(z,t) = d\,z\,\sqrt{K(z,t)}$ is the turbulence diffusivity and d is a positive coefficient.

Finally, we summarize the equations of the present model [Minimalist Model (Min-Model)] for TD-TBL as

$$\nu S(z,t) + \tau(z,t) = \tau_*(t) + \int_0^z dz' \frac{\partial}{\partial t} V(z',t), \tag{4a}$$

$$\Big[\frac{\partial}{\partial t} + \frac{b\sqrt{K(z,t)}}{z} - d\frac{\partial}{\partial z} z\sqrt{K(z,t)}\frac{\partial}{\partial z}\Big] K(z,t) = = \tau(z,t)S(z,t), \tag{4b}$$

$$\tau(z,t) = c^2 K(z,t). \tag{4c}$$

The boundary conditions for at the wall ($z=0$) and at the edge of the TBL ($z = \mathcal{Z}(t)$) are:

$$\text{at the wall } z=0: \ V(0,t)=0\,, \quad K(0,t)=0\,, \tag{5a}$$

$$\text{at } \ z=\mathcal{Z}(t): \ V(\mathcal{Z},t)=V_\infty\,, \ K(\mathcal{Z}(t),t)=0. \tag{5b}$$

The numerical values of the model coefficients b and c are prescribed to be 0.34 and 0.53 respectively to obtain results in agreement with experiments (e.g. [2]), and with DNS data (see Fig. 3 in [3]). The third coefficient d is fixed equal to 0.07 in order to match the role of the turbulent diffusivity in fully-developed turbulent channel flows [4].

We normalize the variables of the TD-TBL in "wall units" using the friction velocity $u_*(t)$, viscous length-scale $\ell_*(t)$, and viscous time-scale $t_*(t)$ defined as

$$u_* \equiv \sqrt{\tau_*(t)}, \quad \ell_* \equiv \nu/u_*, \quad t_* \equiv \nu/\tau_*(t) \tag{6}$$

and denote the normalized variables $V^+ \equiv V/u_*$, $K^+ \equiv \equiv K/u_*^2$, $z^+ \equiv z/\ell_*$, $t^+ \equiv t/t_*$. The friction velocity Reynolds number is defined as $\mathrm{Re}_\tau(t) \equiv u_*\mathcal{Z}(t)/\nu = = \mathcal{Z}(t)/\ell_* = \mathcal{Z}^+(t)$, which is the width of TD-TBL in wall units. It should be noted that a similar normalization is usually employed for the spatially developing TBL (SD-TBL) except that the dependent variables in that case are functions of the streamwise x location instead of time.

The main advantage of the above normalization is that in wall units the *stationary* SD-TBL demonstrate universal behavior. We assume that the *time-developing* TBL exhibits similar universality in the limit of large times, as will be clarified later. We will also show that the time-developing TBL has properties very similar to that of the stationary TBL and its subregions can be classified in the same manner.

"Logarithmic accuracy" for "asymptotically large times". For "asymptotically large times" we mean that $\ln t^+ \gg 1$. We assume that in this limit there is a region in the TBL called the log-layer (with z^+ larger than upper boundary of the buffer sub-layer z^+_{buf}), where the mean streamwise velocity profile is described by the von-Kármán law:

$$V^+(z^+) = \kappa^{-1}\ln z^+ + B\,, \quad z^+ > z^+_{\rm buf} \approx 50, \tag{7}$$

where $\kappa \approx 0.41$ is the von-Kármán constant, and $B \approx 5.2$. We define the edge of the TD-TBL, $\mathcal{Z}(t)$, as the normal distance from the wall where $V = V_\infty$. Thus, the von-Kármán law at $\mathcal{Z}^+(t)$ becomes: $V^+_\infty \equiv \equiv V^+(\mathcal{Z}^+) = \kappa^{-1}\ln\mathcal{Z}^+ + B$. Moreover, in the limit $\ln t^+ \gg 1$ the width of TBL $\mathcal{Z}(t)$ is large enough such that $\ln\mathcal{Z}^+ \gg \kappa B \simeq 1$, thus, $\ln \mathrm{Re}_\tau \equiv \ln\mathcal{Z}^+ \gg 1$. In the next section where we solve Eq. (4), this inequality will allow us to neglect terms of order unity with respect to terms of order $\ln\mathcal{Z}^+$. We thus denote the accuracy of our results as the "logarithmic accuracy". For example, with the logarithmic accuracy, Eq. (7) for V^+_∞ becomes $\kappa V^+_\infty = \ln\mathcal{Z}^+$.

Self-consistent factorized solution. In order to reduce the system of partial differential Eqs. (4) for the unknown functions $S(z,t)$ and $K(z,t)$ of two variables z and t to a system of ordinary differential equations it is convenient to introduce the following non-dimensional functions

$$k(\zeta) \equiv \frac{\tau^+(z,t)}{\tau_*(t)}\,, \ s(\zeta) \equiv \frac{c^3\mathcal{Z}(t)S(z,t)}{b\,\tau_*(t)}\,, \tag{8a}$$

which *in the turbulent region* $z^+ > z^+_{\rm buf}$ are assumed to be only functions of the "outer variable" $\zeta = \zeta(t) \equiv \equiv z/\mathcal{Z}(t)$. Thus Eq. (4c) can be written as

$$K(z,t) = \tau_*(t)k(\zeta)/c^2\ . \tag{8b}$$

We assume also that, the temporal growth of the boundary layer is proportional to the friction velocity,

$$d\mathcal{Z}(t)/dt = \alpha\sqrt{\tau_*(t)}, \tag{8c}$$

with a dimensionless constant α.

We now substitute $K(z,t)$, $S(z,t)$ and $\tau(z,t)$, expressed by Eqs. (8) in terms of $k(\zeta)$, $s(\zeta)$ and $\tau_*(t)$ into the momentum and turbulence energy balance Eqs. (4). We also: i) neglect the viscous contribution νS in Eq. (4a) in the region $z^+ > z^+_{\rm buf}$; ii) account only for the leading contribution to $\partial K/\partial t$ (that originates from $d\mathcal{Z}/dt$) and neglect the contribution of $d\tau_*/dt$. Similarly, in Eq. (2) we account only for the leading contribution in $\ln\mathrm{Re}_\tau \gg 1$ terms. The result is two ordinary differential equations for $k(\zeta)$ and $s(\zeta)$ with explicit expression for $\tau_*(t)$:

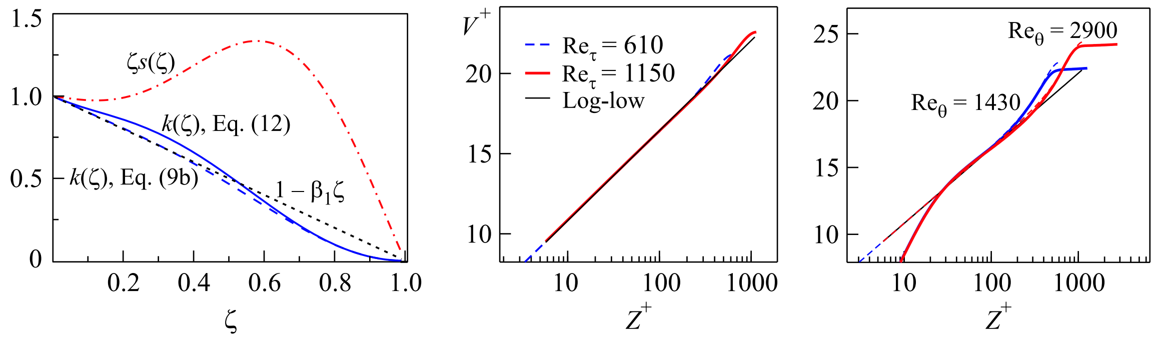

Fig.1. Color online. Left panel: Profiles of approximation to kinetic energy and compensated shear. For discussion see text. Middle panel: Log plots of mean velocity profiles $V^+(z^+)$ in the Min-Model for Re_τ 610 and 1150. Right panel: Comparison of the mean velocity profiles, given by improved Min-Model (dashed lines) and DNS data for $Re_\theta = 1430$, giving $Re_\tau \approx 610$ and for $Re_\theta = 2900$, giving $Re_\tau \approx 1150$. Von-Kármán log-law is also shown in the middle and right panels as straight solid line

$$-\alpha\, c\, \zeta \frac{dk}{d\zeta} + \frac{bk^{3/2}}{\zeta} - d\frac{d}{d\zeta}\zeta\sqrt{k}\,\frac{dk}{d\zeta} = b\,k\,s\,, \tag{9a}$$

$$\kappa\frac{d\,k}{d\,\zeta} + \alpha\,\zeta\, s = 0\,, \qquad \tau_*(t) = \left[\frac{\kappa V_\infty}{\ln(t/t_*)}\right]^2 . \tag{9b}$$

In (9a), the RHS again represent the kinetic energy production, and the three terms in the LHS describe temporal dependence, energy dissipation and the diffusion, respectively. The above system of equations for the functions $k(\zeta)$ and $s(\zeta)$ indicates that the factorization (8a) is consistent *with the logarithmic accuracy* with the momentum and turbulence energy balance (4) in the sense that $k(\zeta)$ and $s(\zeta)$ depend only on one variable ζ.

The boundary conditions for these equations at the edge of TBL, $\zeta = 1$, are:

$$s(1) = 0, \qquad k(1) = 0. \tag{10a}$$

To formulate the boundary condition *near* the wall, it is noted that Eqs. (9) are valid only for $z^+ \geq z^+_{\rm buf} \approx 50$, which corresponds to $\zeta \geq \zeta_{\rm buf} \equiv z^+_{\rm buf}/{\rm Re}_\tau$. In this region the mean velocity satisfies the von-Kármán law (7) which gives $S^+(z^+_{\rm buf}) = 1/\kappa z^+_{\rm buf}$. Noting that the full Min-Model (4) leads in the stationary case to $K^+(z^+_{\rm buf}) = c^{-2}$ and to Eq. (7) with $\kappa = c^3/b$, we obtain with help of Eq. (8a):

$$\zeta_{\rm buf}\, s(\zeta_{\rm buf}) = 1,\ \ k(\zeta_{\rm buf}) = 1. \tag{10b}$$

Since $\zeta_{\rm buf} \equiv z^+_{\rm buf}/{\rm Re}_\tau$, then we take the limit of Eq. (10b) as $\zeta_{\rm buf} \to 0$ for asymptotically large times when $\ln {\rm Re}_\tau \gg 1$. It should be noted that boundary conditions (10b) are formulated *near the wall* (in the log-law region) but not *at the wall*, in the viscous layer, where $S^+ \approx 1$.

With the boundary conditions (10b) and Eqs. (8a) we obtain the mean velocity $V(z,t)$ by simple integration:

$$V(z,t) = \sqrt{\tau_*(t)}\left[B + \frac{1}{\kappa}\int_{{\rm Re}_\tau^{-1}}^{z/\mathcal{Z}} s(\zeta')d\zeta'\right]. \tag{11}$$

Here the lower limit of integration and term B are chosen to satisfy the von-Kármán law, Eq. (7), in the turbulent log-law region (for details, see Appendix).

In order to solve analytically Eq. (8), we now introduce a polynomial form of $k(\zeta)$:

$$k(\zeta) = (1-\zeta)^2(1+\beta_1\zeta+\beta_2\zeta^2+\beta_3\zeta^3+\beta_4\zeta^4)^2, \tag{12}$$

which satisfies the boundary conditions Eq. (10). Substituting (12) into (8) and (10), and neglecting terms of third-order and higher in the resulting equations give the five constants $\alpha\,, \beta_1\,, \dots\ \beta_4$ as

$$\begin{aligned}
&\beta_1 = 1-\beta_0/2, \quad \beta_0 \equiv \alpha/\kappa,\\
&\beta_2 = \beta_0(1+c\kappa/d)/2-(2+\beta_3+\beta_4),\\
&\beta_3 = 3-\beta_0\,(1+13c\kappa\beta_0/6)/2-2\beta_4,\\
&\beta_4 = -4+\beta_0\left[1+c^4(c^4+bd)/24+371c\kappa/(108d)\right]/2.
\end{aligned} \tag{13}$$

Substituting $\kappa = \alpha\int_0^1 \zeta\, s(\zeta)\, d\zeta$ [which follows from integrating Eq. (9b)] into Eq. (13), the constants $\beta_0\,, \dots\ \beta_4$ can be expressed in terms of the model's three parameters b, c and d. For the fixed values of $b = 0.36$, $c = 0.53$ and $d = 0.07$, the resulting values of β_i are $\beta_0 \approx 0.996$, $\beta_1 \approx 0.502$, $\beta_2 \approx 1.99$, $\beta_3 \approx -3{,}05$, $\beta_4 \approx$ ≈ 1.10.

Fig. 1 (left) displays the profile of $k(\zeta)$ (solid line), given by Eq.(12) and the above values of the constants $\beta_0 \dots \beta_4$. Substituting this profile in the energy balance Eq. (9a) we obtain the profile of $\zeta s(\zeta)$, shown by the dot-dashed line. Substitution of the resulting profile

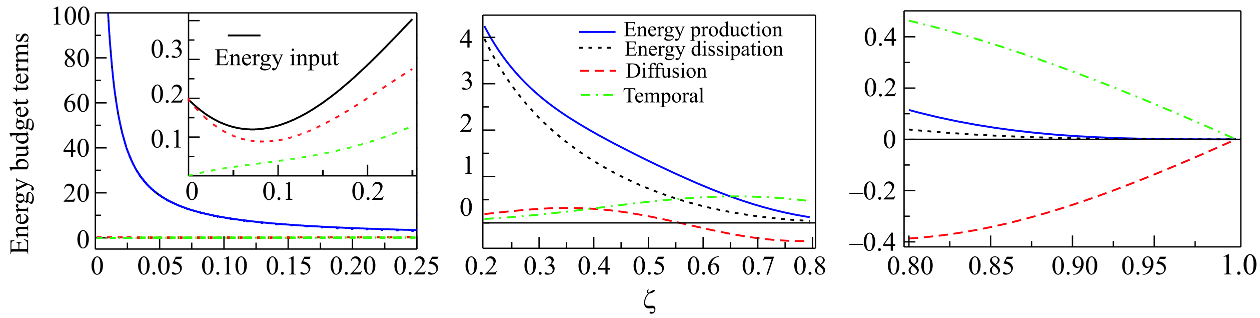

Fig.2. Color online. Energy balance in Eq. (9a). Energy input (the difference between energy production and dissipation) is shown in the insert in the left panel. Line types are same in all panels. Different panels display different regions of TBL

$\zeta s(\zeta)$ into Eq. (9b) gives the profile of $\widetilde{k}(\zeta)$ shown in Fig.1 (left) by the blue dashed line. If the initial function $k(\zeta)$ was the exact solution of the balance Eqs. (8) then the functions $k(\zeta)$ and $\widetilde{k}(\zeta)$ would coincide. Fig.1 shows that these functions are reasonably close. Thus, we conclude that the simple polynomial form (12) approximates the solution of the balance Eqs. (8) in the entire TBL, $0 < \zeta < 1$, with good accuracy.

Comparison of Min-Model results with DNS. In the present section, we compare the results of our Min-Model for TD-TBL with those obtained by DNS of SD-TBL [5 – 7]. This comparison is justified by the fact that TD-TBL and SD-TBL are asymptotically equivalent in the limit of large Re_τ [1].

a. *Mean streamwise velocity.* In our solution the unsteady and diffusion terms in the turbulence energy balance vanish near the wall, for $\zeta \to 0$. Accordingly in this region the mean streamwise velocity $V(z,t)$ satisfies the von-Kármán law (7), which gives [in our normalization Eq. (8a)] $s(\zeta) \to 1/\zeta$. Indeed, in Fig.1 (left), $\zeta\, s(\zeta)$ become constant for small ζ.

The figure also shows that the product $\zeta\, s(\zeta)$ in the wide region $0 < \zeta < 0.6$ exceeds unity, which is the value of $\zeta s(\zeta)$ in the log-law layer. Accordingly, after integration $s(\zeta)$ over ζ, the resulting mean velocity profile $V_3(\zeta)$, shown in Fig.1 (middle) exceeds the log-law level. Thus, the Min-Model clearly demonstrates the *wake contribution*, described by Coles [8] for stationary channel flow. The wake contribution is also seen in our DNS results for the spatially-developing TBL. However, the size of the wake, given by Min-Model Fig.1 (middle) is smaller than that in DNS see Fig.1 (right). This discrepancy originates from the estimate of the outer scale of turbulence ℓ. In the Min-Model ℓ is estimated, following von-Kármán, as the distance to the solid wall z. This assumption is valid only for $z \ll \mathcal{Z}$, otherwise the scale ℓ is affected by the free upper boundary and saturates. Recently [9] the effect of saturation of ℓ (at level of 0.3 of the channel half-width) on the mean velocity was studied in the channel flow. Accounting for the saturation of ℓ in TD-TBL at the level $\approx \mathcal{Z}/2$ (which affects only the shear) improves the calculated velocity profile, Fig.1 (right). The size of the wake, given by Improved Min-Model compares well with that in DNS. This correction does not effect the other results.

b. *Turbulence kinetic energy.* Fig.2 displays ζ-dependence of various terms in the energy balance. One can see in the left panel that close to the wall, $\zeta < 0.25$, the energy production is well balanced by the energy dissipation; the diffusion and unsteady terms play a minor role. They become relevant only for $\zeta > 0.3$ as shown in the middle panel. Thus the region, $\zeta < 0.25$, can be considered as "equilibrium" TBL with local spatial energy balance. For further clarification, we plot in the insert the difference between the energy production and dissipation. The difference is denoted as the *energy input*. This is the part of the energy flux that is required for the temporal development of TBL. Notice that the energy input is finite, while the energy production and energy dissipation themselves diverge as $1/\zeta$ for $\zeta \to 0$, as shown in Fig.2 (left).

The energy input is essential only in the first half of the TBL, $\zeta < 1/2$. As both energy production and energy dissipation become smaller (see Fig.2 (middle and right)) their difference vanishes. The energy input can be totally neglected in the outer tenth of TBL, where $\zeta > 0.9$. The only source of energy here is the turbulent diffusion, which transports energy from the region $\zeta \lesssim 0.5$ to the region $\zeta \gtrsim 0.5$. Thus, the turbulent diffusion leads to increase the width of the TBL in time. Fig.3 (middle) shows good agreement between the analytical (dashed lines) and DNS (solid lines) profiles of the turbulence kinetic energy. The observed discrepancy between the analytical and DNS results for $z^+ < 50$ is expected since our Min-Model was not designed to predict the buffer sub-layer flow.

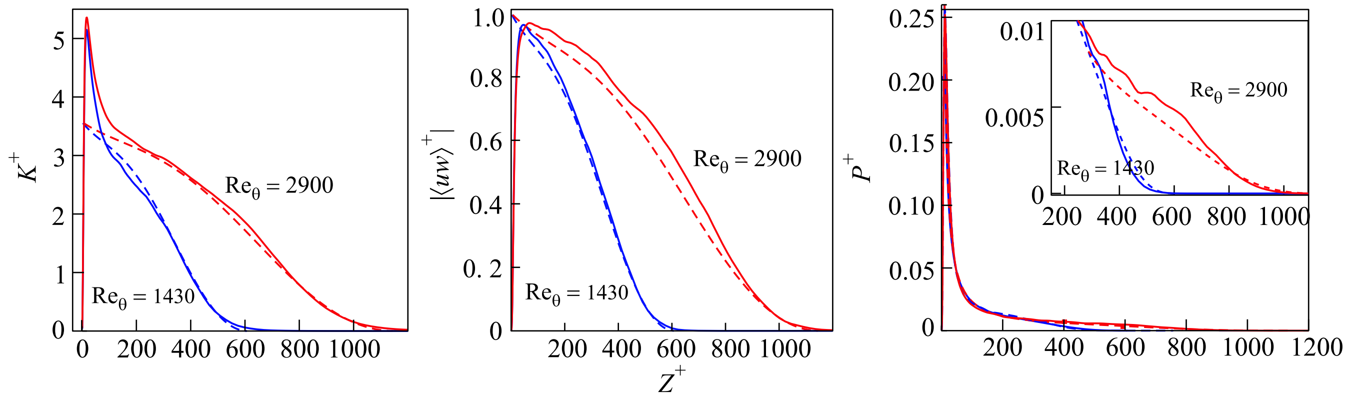

Fig.3. Color online. Comparison of analytical predictions (dashed lines) and DNS results (solid lines) for the turbulence kinetic energy (left), Reynolds stress (middle) and production term (right). Lines corresponding to $Re_\theta = 1430$ (giving $Re_\tau \approx 610$) and to $Re_\theta = 2900$ (giving $Re_\tau \approx 1150$) are marked in the figure. The insert in the right panel shows details of the large z^+ tails

c. Reynolds stress is one of the most important characteristics of TBL, being responsible for the mechanical balance in the turbulent log-law and outer regions. In stationary regimes, the Reynolds stress is prescribed by the outer conditions, maintaining the flow. For example, in pressure driven channel flow (of half-width L) $\tau(z) \propto (1 - z/L)$, while in the zero pressure-gradient plane Couette flow $\tau(z) = \text{const}$. In developing TBL, $\tau(z)$ is not known *a priory*, and thus the Reynolds stress has to be determined self-consistently. Our model [Fig.1(left),$k(\zeta)$] predicts that in the first half of TBL, near the wall $\tau(z) \approx [1 - z/\mathcal{Z}]$, as in the channel flow, while in the last quarter of TBL, near the free boundary, it decays quadratically, $\tau(z) \propto [1 - z/\mathcal{Z}]^2$. This prediction is in a good agreement with the DNS data, Fig.3 (middle). The production term is also well described by our model (see Fig.3, right panel).

Summary. We have presented a simple analytical model (Min-Model) of the physics of time-developing TBL in a Newtonian fluid. The model is based on the exact equation for the momentum flux and on the model equation for balance of turbulence kinetic energy with the production, dissipation and turbulent diffusion terms. The Min-Model results in a partial differential equation for the kinetic energy, which, in the limit of large evolution time was reduced to a relatively simple ordinary differential equation. We obtained an asymptotic formula for the time dependence of width of TBL and approximate analytical solutions for the profiles of the mean velocity, the turbulent velocity fluctuations and the Reynolds stress as a function of time and distance from the wall. These profiles are in good agreement with the DNS observations. In future work we will demonstrate the asymptotic equivalence of temporally- and spatially-developing turbulent boundary layers by the relationship $x \Leftrightarrow V_\infty t$ (here x is the distance from the front edge and V_∞ is the free-stream velocity) and will clarify the difference between these two regimes in pre-asymptotical region, where $\ln t^+$ or $\ln x^+$ are not very large with respect of unity.

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Appendix

Here we describe the derivation of Eq. (11). From the definition of mean shear (1), the mean velocity is written as

$$V(z,t) = V(z_{\text{buf}},t) + \int_{z_{\text{buf}}}^{z} S(z',t)dz'. \qquad (14)$$

Substituting $S(z,t)$ from (8a) into (14) gives

$$V(z,t) = V(z_{\text{buf}},t) + \frac{\sqrt{\tau_*(t)}}{\kappa}\int_{\zeta_{\text{buf}}}^{\zeta} s(\zeta')d\zeta' , \qquad (15)$$

where $\zeta \equiv z/\mathcal{Z}(t)\,,\zeta_{\text{buf}} \equiv z_{\text{buf}}/\mathcal{Z}(t),$. The mean velocity $V(z_{\text{buf}})$ at the edge, (z_{buf}), of the log-law region is given by the von-Kármán law as

$$V(z_{\text{buf}}) = \sqrt{\tau_*}\,[B + \ln z^+_{\text{buf}}]. \qquad (16)$$

Noting that $s(\zeta) = 1/\zeta$ for $\zeta \ll 1$, Eq. (15) becomes

$$V(z,t) = \sqrt{\tau_*(t)}\Big[B + \frac{1}{\kappa}\int_{\zeta_0}^{\zeta} s(\zeta')d\zeta'\Big] , \qquad (17)$$

where $\zeta_0 = \ell_*/Z(t) = 1/\text{Re}_\tau$.

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