EFFECT OF STRONG REPULSION OF COMPOUND PARTICLES (NUCLEI, ATOMS) AT SHORT DISTANCES

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Let us consider the collision of two compound particles consisting of identical fermions (two atoms, two atomic nuclei). It has been known for a long time that the Pauli principle should lead to a strong repulsion between the particles at short distances, and this is always taken into account, for example, when choosing phenomenological potentials for the interaction between the atoms. Repulsion sets in at distances between particles r_{12} such that their wave functions begin to overlap. We shall develop below a quantum-mechanical theory of this effect.

Let the first particle consist of A_1 fermions, let its internal wave function be $\phi_1(\vec{\rho}_1)$, let the radius be R_1, and let its center of gravity be at the point $\vec{r}_1$; $\vec{\rho}_1$ denotes the aggregate of $(3A_1 - 3)$ coordinates describing the relative motion of the fermoins in the particle. Similar symbols A_2, $\phi_2(\vec{\rho}_2)$, R_2, and $\vec{r}_2$ will be used for the second particle.

The wave function of the colliding particles is written in the form

$$\psi = \Phi(\rho)\frac{1}{\sqrt{N}}\hat{A}\left\{z^L Y_{LM}(z)\,\phi_1(\vec{\rho}_1)\,\phi_2(\vec{\rho}_2)\right\} = \Phi(\rho)\,w(\vec{\rho}). \qquad (1)$$

Here $z = (A_1A_2/A)^{1/2}(\vec{r}_1 - \vec{r}_2)$ is the "reduced" distance between particles, L the orbital angular momentum of their relative motion, $\Phi(\rho)$ a function describing the relative motion of the particles. The symbol $\hat{A}$ in (1) denotes the antisymmetrization of the expression in the curly brackets over all $N \equiv A!/A_1!A_2!$ permutations that transform the fermions from one particle into the other and vice-versa.

To describe the relative placements of all $A = A_1 + A_2$ fermions of our system, we have introduced in (1) the vector $\vec{\rho}$ in $(3A - 3)$-dimensional space of the relative coordinates of the fermions. It is convenient to use a spherical coordinate system in this space. The aggregate of $(3A - 4)$ angles will be denoted by $\Omega_{\vec{\rho}}$, and the length ρ of the vector $\vec{\rho}$ is then equal to

$$\rho^2 = \frac{1}{A}\sum_{i>j}^{A}(r_i - r_j)^2 = \sum_{i=1}^{A}(r_i - R)^2 = A\overline{r^2}. \qquad (2)$$

Here $\vec{r}_i$ is the coordinate of the i-th fermion, $\vec{R}$ is the position of the common center of gravity, and $(\overline{r^2})^{1/2}$ is the rms radius of the system. We introduce analogously the quantities ρ_1, Ω_1 and ρ_2, Ω_2 for the first and second particles.

Since for such a definition of the coordinates we have identically

$$\rho^2 = \rho_1^2 + \rho_2^2 + z^2 = A_1\overline{r_1^2} + A_2\overline{r_2^2} + \frac{A_1A_2}{A}r_{12}^2 \qquad (3)$$

and since the first two terms are bounded here (approximately $\overline{r_1^2} = 3R_1^2/5$ and

$$\overline{r_2^2} = \frac{3}{5}A_2R^2 \text{ and } \rho_1^2 + \rho_2^2 = \frac{3}{5}(A_1R_1^2 + A_2R_2^2) = a,$$

where R_i is the radius of the i-th particle), it follows that

$$\rho \approx \left(\frac{A_1 A_2}{A}\right)^{1/2} r_{12} \quad \text{if } r_{12} > R_1 + R_2 . \tag{4}$$

This justifies the representation of the wave function of the entire system in the form (1).

To find the function $\Phi(\rho)$, we substitute (1) in the Schrödinger equation of our system

$$\left(-\frac{h^2}{2m}\sum_{i=1}^{A}\frac{\partial^2}{\partial r_i^2} + \sum_{i>j}^{A} V(ij) - E\right)\Phi(\rho)\, w(\vec{\rho}) = 0, \tag{5}$$

where V(ij) is the interaction between the i-th and j-th fermions, m is the fermion mass, and E is the total energy. Multiplying this equation from the left by $w^+(\vec{\rho})$, integrating with respect to $d\Omega_{\vec{\rho}}$, and using the technique described, for example, in [1], we find for $\Phi(\rho)$ the equation:

$$-\frac{h^2}{2m}\left[\Phi'' + \frac{\mu'}{\mu}\Phi'\right] + [V(\rho) - \epsilon]\,\Phi(\rho) = 0, \tag{6}$$

where the prime denotes differentiation with respect to ρ, $V(\rho)$ is the suitably averaged interaction between the fermions contained in our particles, ε is the energy of the relative motion of the particles, and

$$\mu = \rho^{3A-4}\int d\Omega_{\vec{\rho}}\, |w(\vec{\rho})|^2 . \tag{7}$$

Using the K-harmonics technique (see [1]), we can show furthermore that

$$\mu = \begin{cases} \text{const}\ \rho^{2K_m + 3A - 4}; & \rho \ll \rho_0 \\ \rho^{2(L+1)}; & \rho > \rho_0 \end{cases} . \tag{8}$$

Here K_m is a certain characteristic number introduced in the K-harmonics procedure (for fermions of one type with spin 1/2 we have $K_m \approx (3A)^{4/3}/4$ when $A \gg 1$), and ρ_0 is the value of the coordinate at which the particles "touch" (see [3])):

$$\rho_0^2 = a + \frac{A_1 A_2}{A}(R_1 + R_2)^2 . \tag{9}$$

Introducing a new function $\phi(\rho) = \mu^{-1/2}\Phi(\rho)$ we obtain in place of (6) the simple equation

$$-\frac{h^2}{2m}\phi'' + [V(\rho) + U(\rho) - \epsilon]\phi = 0, \tag{10}$$

which has the form of a radial Schrodinger equation for a particle in a field. The potential that has appeared here

$$U(\rho) \equiv \frac{\hbar^2}{2m}\frac{1}{2\sqrt{\mu}}\left(\frac{\mu'}{\sqrt{\mu}}\right)'$$

describes the particle repulsion resulting from the Pauli principle. Using (8), we get

$$U(\rho) = \begin{cases} \dfrac{\hbar^2}{2m}\dfrac{\mathcal{L}(\mathcal{L}+1)}{\rho^2}; & \rho \ll \rho_0 \quad (11a) \\ \dfrac{\hbar^2}{2m}\dfrac{L(L+1)}{\rho^2}; & \rho > \rho_0 \quad (11b) \end{cases}.$$

Here $\mathcal{L} = K_m + 3(A - 2)/2$ is a large number closely connected with the total kinetic energy A of the fermoins: the expression in the upper line of formula (11) is none other than the minimum value of the kinetic energy A of the fermoins contained in the sphere $|\vec{\rho}| = \rho$.

The behavior of $U(\rho)$ at $\rho \simeq \rho_0$ can be understood by choosing for $\mu(\rho)$ some convenient interpolation formula, for example

$$\mu = \rho^{2(L+1)}\frac{x^\sigma}{x^\sigma + x^{2(L+1)}}; \quad x \equiv \frac{\rho}{\rho_0}; \quad \sigma \equiv 2K_m + 3A - 4. \tag{12}$$

We then obtain immediately

$$U(\rho) = \frac{\hbar^2}{2m\rho^2}\left\{L(L+1) + \frac{\Gamma(L+1)}{1+x^\Gamma} + \frac{\Gamma}{4}\,\frac{(\Gamma-2) - 2(\Gamma+1)x^\Gamma}{(1+x^\Gamma)^2}\right\}, \tag{13}$$

where $\Gamma \equiv \sigma - 2(L + 1)$.

In practically all cases $K_m \gg 1$, and all the more $\sigma \gg 1$. At not too large values of L, when $\Gamma \gg 1$, expression (13) can be easily investigated. It turns out here that formula (11a) is valid when $\rho \leq \rho_0[1 - (\ln\Gamma)/\Gamma]$, and formula (11b) is valid when $\rho \geq \rho_0[1 + (\ln\Gamma)/\Gamma]$.

If both colliding particles are sufficiently "hard," i.e., they have almost define values of ρ_1 and ρ_2 (this corresponds to small fluctuations of the rms radius), then ρ and the distance r_{12} between the particles are uniquely related by formula (3). Combining (3) with (11) - (13) we then obtain for the radius of the repelling core

$$(r_{12})_{core} = (R_1 + R_2)\left[1 - \frac{A}{A_1A_2}\,\frac{\rho_0^2}{(R_1+R_2)^2}\,\frac{\ln\Gamma}{\Gamma}\right] \approx (R_1 + R_2), \tag{14}$$

and its value

$$U_{core} \approx \frac{\hbar^2}{2m}\frac{\mathcal{L}(\mathcal{L}+1)}{\rho^2} \tag{15}$$

as already mentioned, coincides with the kinetic energy A of the fermions contained inside the spherical volume with the radius $R \approx \rho_0/\sqrt{A}$ (ρ_0 is calculated with the aid of (9)).

The foregoing analysis shows that two particles made up of fermions of one type will always repel each other as soon as the distance between them becomes smaller than their total radius. At not too high collision energies, the particles will not be able to penetrate into each other at all. In atomic physics this phenomenon has been known for a long time. In nuclear physics, an analogous phenomenon should be observed in scattering of particles by each other.

[1] A. Baz' and M. Zhukov, Yad. Fiz. 11, 779 (1970) [Sov. J. Nucl. Phys. 11, 435 (1970)].

N O T E

For technical reasons, the balance of the Russian version of Volume 14, Number 11 will be published in Volume 14, Number 12 of the translation.