$H_x = 0$ in the transition layer. Owing to the smallness of $\bar{\chi}$, Eq. (5) does not have in this case nonvanishing slowly varying solutions. Therefore, to calculate the domain wall it is necessary to use Eq. (4), in which the field H is set equal to H_k.

Putting for simplicity $4\pi M(B) = a \sin k\tilde{B}$, where $\tilde{B} = B - [(B_1 + B_2)/2]$ and $ak - 1 = \kappa^2 \ll 1$, we obtain the simple equation

$$-\kappa^2 \tilde{B} + \frac{k^2 \tilde{B}^3}{6} = \frac{r_0^2}{4} \frac{d^2 \tilde{B}}{dy^2}, \quad \tilde{B}(\pm\infty) = \pm \frac{\kappa\sqrt{6}}{k},$$

the solution of which is

$$\tilde{B}(y) = \frac{\kappa\sqrt{6}}{k} \operatorname{th} \frac{y}{2d}; \quad d = \frac{r_0}{2\sqrt{2}\kappa}. \tag{6}$$

The surface tension Δ is equal to

$$\Delta = \frac{1}{4\pi} \int_{-\infty}^{\infty} dy \left[\int_{B_1}^{B(y)} (H_0(B) - H_k)\, dB + \frac{r_0^2}{8} \left(\frac{\partial B}{\partial y} \right)^2 \right] = \frac{1}{24\pi} d\left(\frac{\partial H}{\partial B} \right)_{1,2} \times (B_2 - B_1)^2. \tag{7}$$

In the case when $B_2 - B_1$ is small compared with the period of the oscillations, including also the limiting case $(\partial M/\partial B)_{max} \gg 1$, the dimension of the domain wall is $d \sim r_0$. In this case (7) gives the correct order of magnitude of the surface energy.

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PERIODIC MAGNETIC STRUCTURES AND PHASE TRANSITIONS

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We shall demonstrate, using the electron diamagnetic moment as an example, a new type of phase transition (from a homogeneous structure to a periodic one), the necessary condition for which is a nonlocal connection between the thermodynamic quantities, and for which it is possible to explain the character of the singularity at the transition point. The diamagnetic moment $\vec{M}$ is determined [1,2] by the values of the magnetic induction* $\vec{B}$ over the entire Larmor orbit of radius r, and (F_m - proper free energy of the magnet)

$$\mathbf{M} = -\delta F_m / \delta \mathbf{B}, \quad \mathbf{B} = \mathbf{H} + 4\pi \mathbf{M} \tag{1}$$

are integral equations which, generally speaking, have non-growing oscillating solutions $\vec{M} = \vec{M}(\vec{r})$. (We choose the z axis in the direction of $\vec{H}$; if the electrons have an infinite mean free path, there is no dependence on z, $\vec{r} = (x, y)$, and $\vec{H}$ is a constant vector, since curl $\vec{H} = 0$.) The "usual" equilibrium homogeneous $\vec{M}_0$ can become inhomogeneous when the external parameters ($\vec{H}$ and the temperature T) are varied, for two reasons.

1. The total free energy

$$F_t = -\frac{1}{4\pi}\int \mathbf{B}\, d\mathbf{H} = F_m + \int\left(\frac{H^2}{8\pi} + 2\pi M^2\right) d\mathbf{r} = F_m + \int\left\{\frac{H^2}{8\pi} - \frac{1}{2}\mathbf{M}\frac{\delta F_m}{\delta \mathbf{M}_1}\right\} d\mathbf{r} \equiv \tag{2}$$

$$\equiv \int f_t\, d\mathbf{r};\ \mathbf{M}_1 = \mathbf{M} - \mathbf{M}_0,$$

becomes smaller in the inhomogeneous state than in the homogeneous one. Near the transition point T_1 we have $M_1 \ll M_0$ and (taking into account the translational and central symmetry and including, for example, the Fermi-liquid interaction)

$$F_m = F_m^{(0)} + \int \mathbf{K}_1(\mathbf{r} - \mathbf{r}')\mathbf{M}_1(\mathbf{r}')\, d\mathbf{r}' + \frac{1}{2}\int\int K_2^{ij}(\mathbf{r} - \mathbf{r}', \mathbf{r} - \mathbf{r}'')\, M_1^i(\mathbf{r}')\times$$

$$\times M_1^j(\mathbf{r}'')\, d\mathbf{r}'d\mathbf{r}'' + \ldots, \tag{3}$$

where K_i are even functions of all their arguments and $\vec{M}_0 = -(1/4\pi)\int\vec{K}_1(\vec{r})d\vec{r}$. Equations (3) and (1), together with div $\vec{M}_1 = 0$, determine the periods of the resultant structure. Thus, if z coincides with one of the principal crystallographic axes (for simplicity we shall consider only this case) and $M_x = M_y = 0$, then in the principal approximation

$$M_1 = M_{1z} = A\cos(kx)\cos(ky);\ \int_{-\infty}^{\infty}\int L_1(x,y)\cos(kx)\cos(ky)\,dx\,dy = \tag{4}$$

$$= L_1(k) = 1,$$

$$L_1(\mathbf{r}) = \int K_2(\mathbf{r} - \mathbf{r}', \mathbf{r}')\, d\mathbf{r}'. \tag{4a}$$

Since $L_1(0) = 4\pi\chi < 1$ (the condition of thermodynamic stability for $\vec{M}_0$; $\chi = \partial M_0/\partial B_0$) the sign of $L_1(r)$ must alternate for k to be real (there is obviously an infinite number of complex k) and this in turn calls for the presence of more than one zone (this is not obligatory for finite A). The solution obtained in [3] for $A \to 0$ in the case of one zone, with $4\pi\chi > 1$, is not thermodynamically stable. The next approximations lead to the appearance of additional harmonics and to the modulation of the oscillations with a period proportional to A^{-2}. (Thus, in the one-dimensional case $M_1 = \mathrm{Re}\,\Sigma_{m=1}^{\infty} A^m\, \Sigma_{\ell=0}^{m}\, a_{m\ell}(A^2y)\,\exp(i\ell ky)$; $a_{10} = 0$; all the functions a_{ml} are linearly independent to the exclusion of solutions that increase with y in the next higher approximations.) The expansion of F_t in powers of A (which can be obtained by simply substituting (4) in (2) and in (3), the latter being written out up to sixth order in M_1 inclusive), has near T_1 the same form as the Landau expansion [4], but starts with A^4: $F_t = F_t^0 + a(T - T_1)A^4 + bA^6 + \ldots$ and leads (after minimization with respect to A) to a third-order transition with a relative jump of the order of Δ in the derivative of the electronic specific heat with respect to the temperature. ($M_1 = 0$, and therefore it is immaterial whether the independent variable in (2) is H or B.)

2. Since L_i is even and therefore (4) gives a pair of solutions $\pm k$, it is of interest to investigate the degeneracy point $k = 0$ ($T = T_2$), near which (see (4)) $4\pi\chi \to 1$ and M_1 varies infinitely slowly. Using this, we expand f_t (see (2)) in powers of dB/dy (for simplicity, we denonstrate the solution for the one-dimensional case **: $f_t = f_t^0(B) - (1/2)\varphi B'^2$, with B determined by the minimum of F_t (thus, (1) follows from the minimization of (2)). This means that f_t differs only in sign from the Langrange function of one-dimensional particle motion (with B, y, φ, and $f_t^0(B)$ playing respectively the role of the coordinate, time, mass (of arbitrary sign) and potential energy) - a problem whose solution is well known (see [7]; thus, $y - y_0 = \pm\int[2\alpha(\beta - f^0(B))]^{-\frac{1}{2}}dB$; β = const - "total energy"). In the homogeneous case $\partial f_t^0/\partial B_0 = 0$ and since $\partial^2 f_t^0/\partial B_0^2 = 0$ when $4\pi\chi = 1$, we must have $\partial^3 f_t^0/\partial B_0^3 = 0$ and $\partial^4 f_t^0/\partial B_0^4 > 0$ for the minimum of f_t^0. This determines, by the same token, B_0, T_2, and $H_0 = B_0 - 4\pi M(B_0)$.

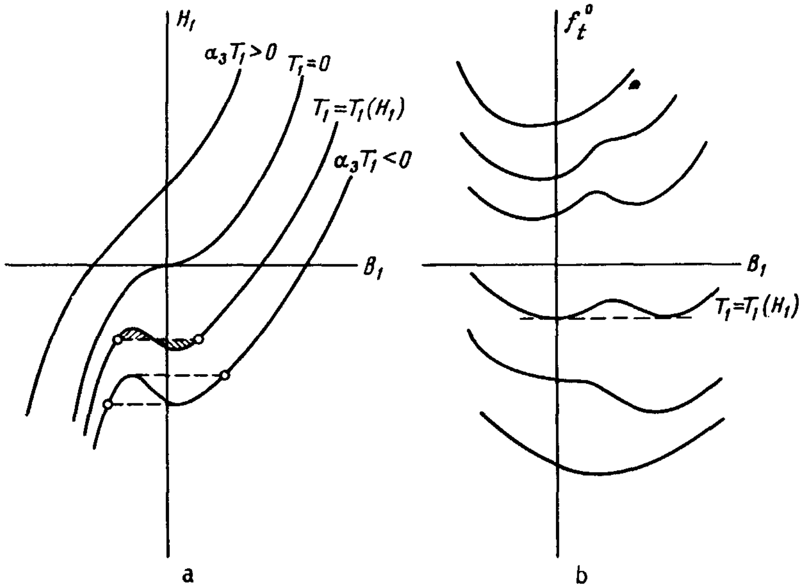

Fig. 1

Expanding $f_t^0 = (B - H)^2/(8\pi) + f_m^0(B) + (H^2/8\pi)$ in powers of $B_1 = B - B_0$ and $T_1 = T - T_2$, we get $f_t^0 = f_t^0(B_0, T_0) + \alpha_1 T_1 + (\alpha_2 T_1 - H_1/4\pi)B_1 + (1/2)\alpha_3 T_1 B_1^2 + (1/4)\alpha_4 B_1^4$; $\alpha_4 > 0$. From Fig. 1 (where $H_1(B_1)$ is obtained from $\partial f_t^0/\partial B_1 = 0$; for concreteness α_2 and $\alpha_3 < 0$; the circles denote solutions that are stable for a specified H in the homogeneous case; the curves are similar to the $-p = -p(v)$ curves near the liquid-vapor critical point), it is clear that the phase transition (when $H_1 \leq 0$) corresponds either ($H_1 < 0$) to the case considered in item 1, or ($H_1 = 0$) - even in the homogeneous case - to a transition of the "4/3" type when $T_1 = 0$, i.e., $F_t \simeq -T_1^{4/3}$, and the specific heat is $c_v \simeq T_1^{-2/3}$ (in the case of one zone, when $\alpha_2 = 0$ - to a second-order transition). We call attention to the "domain solution" which arises (if the "mass" $\varphi < 0$) when $T_1 = T_1(H_1)$ (with $f_t(B_1) = f_t(B_2)$, since $B_1' = B_2' = 0$), and which was first indicated in [8], and to the fact that the maximum amplitude of the oscillation is bounded when $\varphi < 0$.

3. The transition from the homogeneous to the inhomogeneous case can be connected with the occurrence (when $T = T_3$) of proper solutions of Eqs. (2) and (3) when the energy connected with the inhomogeneity is already essentially negative. Since growing solutions (with "complex wave vector") exist for real equations only in pairs, the necessary condition for the transition point is double degeneracy. It is easy to verify that the eigenfunctions can be produced with infinitesimally small amplitude $A \sim (|T_1|_3)^{1/2} T_1 = T - T_3$. The solution, in analogy with the one given in item 1, should be taken only in the zeroth approximation in the form (4) with $T = T_3$ (when $k = k_0$ and $L_1'(k_0) = 0$), and it must be recognized that (in dimensionless units) $A|T_1| \sim A^3 \sim A''$. On going through T_3 we have $\delta F_t \sim A^4 \sim T_1^2$ and a second-order phase transition takes place with a Landau jump of the specific heat [4].

Let us consider, finally, the case when the eigenfunctions are produced at $T = T_0$ directly with a finite amplitude A_0 and a period λ_0; $M_1|_{T=T_0} = f_0(\vec{r})$ (see Fig. 2, which shows the domain in A and T in which eigenvalues $\lambda = \lambda(A, T)$ exist; in the general case A is an arbitrary constant of the equation, which goes over into the amplitude when $A \to 0$).

In an infinite sample this would lead to a jump of F_t, meaning to a negative specific heat c_v, a state which is absolutely unstable. We must therefore take into account magnetostriction and use the thermodynamic potential $\Phi(pT)$. When $A \neq 0$ the phase transition is isomorphic and of first order (the relative adiabatic change in temperature due to the latent heat is of the order of Δ), since $T_0 = T_0(v)$, and the periodic solution continues until the values of Φ coincide in the homogeneous and inhomogeneous states. The specific volume v (i.e., the period of the crystal lattice) and the electron density are also periodic and have the same period as the magnetic structure: $v(r) = \delta\Phi/\delta p$ (p - pressure).

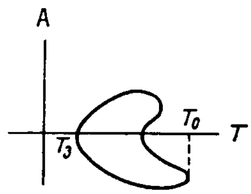

Fig. 2

Mathematically, the problem of determining M_1 reduces to a solution in the vicinity of T_0 of the functional equation $\hat{L}\{T_0 + T_1; M1\} = \hat{L}_0\{M_1\} + T_1\hat{L}_1\{M_1\} = 0$. We shall seek M_1 in the form

$$M_1(r) = f(r; \lambda_0 + \lambda_1, T_0 + T_1) = f_0(r) + \lambda_1 g(r) + T_1 h(r) + 1/2\,\lambda_1^2 q(r) = \\ = f_0(r) + \lambda_1 g(r) + H(r). \tag{5}$$

Of course, such an expansion in terms of λ_1 can be used, strictly speaking, only in regions of r which are small compared with λ_1^{-1}; however, since the nonlocalities connected with distances of the order of the Larmor radius, and $\lambda_1^{-1} \to \infty$, this condition is sufficient, and it is only necessary to "invert" the resultant solution to $f(r; \lambda_0 + \lambda_1, T_0 + T_1)$. Substituting the expansion (5) into the equation and stipulating that there is no term linear in λ_1, we get $g(r) = A_1 G(\vec{r})$, where $G(\delta\hat{L}_0/\delta f_0)(\vec{r}) = 0$. Equality of the periods in $G(\vec{r})$ and $f_0(\vec{r})$, namely $\lambda^{(f_0)}(T, A) = \lambda^{(G)}(T)$, i.e., over the eigenvalues of the corresponding equations, yields $T = T(A)$, and the extremal points $T(A)$ yield T_0, A_0, and λ_0.

The function H(r) is determined from

$$\frac{\delta \hat{L}_0}{\delta f_0} H(r) = -T_1 \hat{L}_1\{f_0\} - \frac{1}{2} A_1\lambda_1^2 \frac{\delta^2\hat{L}_0}{\delta f_0^2} G(r) \quad . \tag{6}$$

Since the corresponding homogeneous linear equation which determines G has a nontrivial solution, it is necessary that the right side of (6) be orthogonal to the solution of the corresponding transposed homogeneous equation. This determines the connection between λ_1, T_1, and A_1, i.e., $\lambda_1 = \lambda_1(T_1, A_1)$, and $\lambda_1 = 0$ yields $A_1^{max} = A_1^{max}(T_1)$.

All the foregoing transitions and structures are connected with oscillating $M(B_0)$ and are periodic in H. Of course, to observe them experimentally it is necessary that the external magnetic field be stable in time and that the mosaic structure of the crystal be weak. Using the foregoing results, we can easily obtain also the true "equation of state" of the magnet - a single-valued function $M = M(H, T; \vec{r})$ in the entire region of H and T. Since the origin of M_0 is immaterial, these should hold also for ferromagnetic metals.

In alternating fields of frequency ω, with $\omega\tau \ll 1$ (τ - electron mean free path time), the appearance and vanishing of a periodic structure leads to singularities in the high-frequency characteristics of the magnet.

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* In a typical field-theory problem (charges in vacuum), we have here "self averaging" of H_{micro} over the orbits of different electrons, the distances between which are $a \ll r$, $a/r \sim N^{-2/3}$, and N is in Fermi statistics of the order of the particle number. We note that the inhomogeneity leads (owing to the constancy of the chemical potential) to the appearance of an electric field with potential $\varphi \sim e^{-1}H^2\chi^2a^3\Delta$, where χ is the susceptibility, $\Delta \sim (e\hbar H/cS_{ext})^{1/2}$, and S_{ext} is the extremal area of the Fermi-surface section. We emphasize that the effects considered below are possible also when $(\partial H/\partial B)_{min} = 1 - 4\pi\chi_{max} > 0$.

** A general formula which yields all the K_i and L_i (and which can be conveniently derived by using quantization [5]) has, in the principal quasiclassical approximation, the form $M = \sum_a \langle M^a\{\langle B^{-1}[y + (cp_x - cp_x')/eB]\rangle_a\}\rangle_a$; $\langle h\rangle_a = 2/S_a^{ext}\int p_y^a(p_x)h(p_x)dp_x$, a - number of zone, p - quasimomentum, $\epsilon(p) = \epsilon$, ϵ - energy, $M^a(B^{-1})$ - homogeneous moment from [6]. A periodic structure exists when $k \to 0$ if $\varphi^{-1} = -(1/2)(c/eB_0)^2\sum_a S_{ext}^a\chi_a p_{xa}^2 > 0$.