SOUND IN A DEGENERATE SOLUTION OF He^3 IN SUPERFLUID HELIUM

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It is well known that solutions of He^3 in He^4 become stratified at temperatures below 0.8°K. As $T \to 0$, one end of the stratification curve lies at the point of the pure matter (He^3), and the other at the point corresponding to a solution with concentration of approximately 6% [1]. The properties of the latter are of considerable interest, since when $T \ll T_{deg} = m^* v_F^2/2$ we have a degenerate solution in a superfluid liquid. All the thermodynamic properties of such solutions, owing to the negligible role of the phonons and rotons, are determined by impurity excitations of the Fermi-liquid type. This is confirmed by measurements of the specific heat [2], which varies linearly with the temperature.

We consider here the properties of sound in a degenerate solution near absolute zero, when the collisions of the impurity excitations are unimportant. In this case the excitation distribution function n satisfies a kinetic equation with zero collision integral [3]:

$$\frac{\partial n}{\partial t} + \frac{\partial n}{\partial \mathbf{r}} \frac{\partial H}{\partial \mathbf{p}} - \frac{\partial n}{\partial \mathbf{p}} \frac{\partial H}{\partial \mathbf{r}} = 0, \tag{1}$$

where H is the Hamiltonian function for an excitation with momentum $\vec{p}$ moving in a superfluid liquid having a velocity v_s [4]

$$H = \epsilon + \mathbf{p}\mathbf{v}_s \frac{\delta m}{m^*} - \frac{m v_s^2}{2} \frac{\delta m}{m^*}, \quad \delta m = m^* - m \tag{2}$$

and ϵ is the excitation energy in a reference frame in which the superfluid liquid is at rest. The energy ϵ depends on the momentum $\vec{p}$ and on the density $\rho_i = N_0 n_4$ of the isotope He^4, and is a functional of the distribution function n. We thus have

$$\epsilon = \epsilon_0 + \frac{p^2}{2m^*} + \int f(\mathbf{p}, \mathbf{p}') \delta n(p') d\tau' \tag{3}$$

δn is the deviation of the distribution from equilibrium, and $f(\vec{p},\vec{p}')$ is a function characteristic of the Fermi liquid, describing the interaction of the excitations.

The motion of the superfluid liquid is determined by two equations [5], namely the continuity equation

$$\dot{\rho} + \operatorname{div}(\rho_1 \mathbf{v}_s + \int \mathbf{p} n \, d\tau) = 0 \tag{4}$$

and the equation of motion of the superfluid liquid

$$\dot{\mathbf{v}}_s + \nabla \left\{ \frac{v_s^2}{2} + \mu_0(\rho_1) + \int n \frac{\partial H}{\partial \rho_1} d\tau \right\} = 0. \tag{5}$$

Equation (5) can be derived in the same manner as in pure superfluid helium, by starting from the momentum conservation law [5]. It is only necessary to take additional account of the contribution made to the momentum-flux tensor Π_{ik} by the Fermi-liquid effects

$$\delta \Pi_{ik} = \int n d\tau \int f(\mathbf{p}_1 \mathbf{p}') \delta n(p') d\tau'. \tag{6}$$

The oscillatory motion of the solution is described by the linearized system of equations (1), (4), and (5), in which v_s and the small increments ρ_1' and n_1 vary like $\exp(i\omega t - ikx)$. From the foregoing equations we get ($\nu = \cos \nu$)

$$(\omega - k\frac{\partial \epsilon}{\partial p}\nu) n_1 + \frac{\partial n_0}{\partial \epsilon} k \frac{\partial \epsilon}{\partial p} \nu (\frac{\partial \epsilon}{\partial \rho_1}\rho_1' + p v_s \nu + \int f n_1 d\tau) = 0; \tag{1}$$

$$\omega v_s - k[(\frac{s^2}{\rho_1} + \int n \frac{\partial^2 \epsilon}{\partial \rho_1^2} d\tau) \rho_1' + \int n_1 \frac{\partial \epsilon}{\partial \rho_1} d\tau] = 0; \tag{4}$$

$$\omega(\rho_1' + m\int n_1 d\tau) - k(\rho_1 v_s + \int n_1 p \nu d\tau) = 0 \tag{5}$$

(s - speed of sound in pure superfluid helium).

We shall henceforth assume for simplicity that the function $f = f_0$ does not depend on the directions of the momenta and is constant. Then the condition for the compatibility of the system (1), (4), and (5) is obtained by equating the corresponding determinant to zero. Thus we get the dispersion equation

$$\begin{aligned}(1 - F_0 \omega_1)[u_1^2(1 + c\beta)(1 - c\frac{\delta m}{m^*}) - u^2] + 3w_1 c \frac{m_4}{m^*}[\{(\alpha + \frac{\delta m}{m_4})^2 + \\ + c\beta(\frac{\delta m}{m_4})^2\}u^2 + \alpha^2\{u_1^2(1 - c\frac{\delta m}{m_4}) - u^2\}] = 0,\end{aligned} \tag{7}$$

in which we have introduced the following symbols for the dimensionless quantities:

$$u_1^2 = \frac{s^2}{v_F^2}, \quad u = \frac{\omega}{k v_F}, \quad F_0 = f_0(\frac{\partial \tau}{\partial \epsilon})_{\epsilon=\epsilon_F}, \quad \alpha = (\frac{\partial \epsilon}{\partial \rho_1}\frac{\rho_1}{m_4 s^2})_{\epsilon=\epsilon_F},$$

$$\beta = (\frac{\partial^2 \epsilon}{\partial \rho_1^2}\frac{\rho_1^2}{m_4 s^2})_{\epsilon=\epsilon_F}.$$

Here $c = N_3/N_4$ is the concentration of the solution, N_3 the number of He^3 atoms, m_4 the mass of the He^4 atom, $v_F^2 = \hbar^2/m^{*2}(3\pi^2 N_3)^{2/3}$ and $w_1(u) = -1 + (u/2)\ln[(u+1)/(u-1)]$.

Inasmuch as $u_1^2 \gg 1$, Eq. (7) has a solution with a value of u^2 close to u_1^2. Solving this equation under the assumption that $u^2 \gg 1$ we get

$$(\frac{\omega}{k})^2 = u_1^2 v_F^2 = s^2\{1 + c[\frac{m_4}{m^*}(\alpha + \frac{\delta m}{m_4})^2 + \beta - \frac{\delta m}{m_4}]\}. \tag{8}$$

Formula (8) determines the speed of sound in the degenerate solution as a function of the solution concentration. Estimates show the second term in (8) to be several per cent of s, i.e., to be a perfectly observable effect.

Equation (7) describes also the collective oscillations of the Fermi liquid made up of the atoms of the dissolved isotope He^3, which are analogous to zero sound. Allowance for the terms combined in the second square bracket of (7) is essential here, since, as we shall show, F_0 is of the same order of smallness as cu_1^2 The sought root of (7) should obviously lie near $cu_1^2 = 1$, for only then do we get $w_1(u) \gg 1$ and Eq. (7) can be satisfied. When $u \approx 1$, Eq. (7) takes the form

$$1 - \tilde{F}_0 w_1(u) = 0,\ \tilde{F}_0 = F_0 - 3c\frac{m_1}{m^*}\left[a^2 u_1^2 + \left(a + \frac{\delta m}{m_4}\right)^2\right], \tag{9}$$

so that we get the equation for zero sound, well known from the Fermi-liquid theory, in the approximation in which $F = F_0$ [6]. We see, however, that in the case of a degenerate solution we have renormalization of the constant F_0. This renormalization is due to the Fermi-excitation interaction resulting from the fact that the excitations are in a vibrating superfluid liquid.

Thus, the constant F_0 describes the interaction of Fermi excitations in a solution at standstill. As shown in [4], for a weak solution of isotopes, F_0 can be obtained from simple thermodynamic considerations, and is equal to $3c(m_4/m^*)u_1^2(2a - 1)$. (Unfortunately, the accuracy of this relation is not perfectly clear.)* In this case, the effective value $\tilde{F}_0$ has a negative sign and no undamped zero sound can propagate in the degenerate solution. Inclusion of the first harmonic F_1, as shown by an estimate given in [4], does not change this result qualitatively (owing to the smallness of F_1). This, however, leaves still open the question of the possibility of more complex types of collective oscillations with $\int n_1 d\tau = 0$.

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* The second term in the square bracket of (9) must be taken into account, since a strong compensation of the first term in the same bracket with F_0 takes place ($\alpha = 1.28$ according to [4]).