ENERGY SPECTRUM OF A FINITE FERMI SYSTEM

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We call attention in this note to the possible appearance of a continuous energy spectrum in a system of interacting fermions contained in a finite volume.

Let us consider a system of one-dimensional fermions and antifermions interacting in a box of length L and described in the Schrödinger representation by a Hamiltonian of the form [1]

$$H_g = H_0 + 2gH_1 = -i\int_{-L/2}^{L/2} dx : \Psi_1^*(x)\frac{\partial\Psi_1(x)}{\partial x} - \Psi_2^*(x)\frac{\partial\Psi_2(x)}{\partial x} : + \tag{1}$$

$$+ 2g\int_{-L/2}^{L/2} dx \int_{-L/2}^{L/2} dy : \Psi_1^*(x)\Psi_1(x)V(x-y)\Psi_2^*(y)\Psi_2(y):$$

here $V(x) = V(-x) = V^*(x)$ is the form factor of the interaction*; the field $\Psi_\alpha(x)$ and its Hermitian adjoint $\Psi^*_\alpha(x)$ satisfy the periodicity condition $\Psi_\alpha(-L/2) = \Psi_\alpha(L/2)$; $\Psi^*_\alpha(-L/2) = \Psi^*_\alpha(L/2)$ and are expressed in usual fashin in terms of the fermion and antifermion creation and annihilation operators ** $a^*(p)$, $a(p)$ and $b^*(p)$, $b(p)$

$$\Psi_\alpha(x) = \frac{\sqrt{2\pi}}{L}\sum_q e^{-iqx}\Psi_\alpha(q) \equiv \frac{\sqrt{2\pi}}{L}\sum_q e^{iqx}[u_\alpha(q)a(q) + v_\alpha(q)b^*(-q)]; \tag{2}$$

where $\alpha = 1, 2$; $-L/2 \le x \le L/2$; the operators $a(p)$ and $b(p)$ are normalized so that

$$\{a(p),\ a^*(q)\}_+ = \{b(p)b^*(q)\}_+ = \frac{L}{2\pi}\delta_{p,q} = \begin{cases} L/2\pi, & p = q \\ 0, & p \neq q \end{cases} \tag{3}$$

and

$$u_1(q) = v_2(q) \equiv \theta(q \geqslant 0) = \begin{cases} 1, & q \geqslant 0 \\ 0, & q < 0 \end{cases};\quad u_2(q) = v_1(q) \equiv \theta(q<0) = \begin{cases} 0, & q \geqslant 0 \\ 1, & q < 0 \end{cases};$$

The symbol :...: in (1) denotes the usual Wick ordering in $a^*(p)$, $b^*(p)$; $a(p)$, $b(p)$. The states of the physical system under consideration will be described by vectors in a Fock space H for the operators a^*, a, b^*, and b.

The Hamiltonian H in (1) defineds for arbitrary** g a self-adjoint operator in H. However, the character of the spectrum of this operator depends strongly on the quantity $|2gV(p)|$. When $|2gV(p)| < 1$ it is positive-definite and discrete; when $|2gV(p)| = 1$ the spectrum of H is bounded from below but contains continuous branches; when $|2gV(p)| > 1$ the spectrum of H contains continuous branches filling the entire real line.

To verify this, we introduce (for $p \neq 0$) operators defined on a set of finite vectors, $\tilde{\mathsf{H}}$, which is dense in H

$$A(p) = \frac{2\pi}{L\sqrt{|p|}}[\theta(p>0)\sum_q : \Psi_1^*(q)\Psi_1(q+p): + \theta(p<0)\sum_q : \Psi_2^*(q)\Psi_2(q+p):] \tag{4}$$

$$A^+(p) = \frac{2\pi}{L\sqrt{|p|}}[\theta(p>0)\sum_q : \Psi_1^*(q)\Psi_1(q-p): + \theta(p<0)\sum_q : \Psi_2^*(q)\Psi_2(q-p):].$$

The operators $A(p)$ and $A^+(p)$ are not bounded, but $\tilde{H}$ is invariant with respect to them, so that the commutator $[A(p), A^+(p)]$ has a clearly defined meaning. Direct calculation yields [1]:

$$[A(p), A^+(p)]_- = \frac{L}{2\pi}\delta_{p,q}, \tag{5}$$

meaning that (4) is a representation of Bose commutation relations (BCR). This representation is redducible. It can be proved that the entire space H can be decomposed into a direct sum of subspaces $H(\lambda_1, \lambda_2)$, in each of which the operators $A(p)$ and $A^+(p)$ generate an irreducible BCR representation. The subspaces $H(\lambda_1,\lambda_2)$ are characterized by the fact that they are proper subspaces of commuting operators Λ_1 and Λ_2:

$$\Lambda_a = \frac{2\pi}{L}\sum_q [u_a(q)\, a^*(q)\, a(q) - v_a(-q)\, b^*(q) b(q)]$$

$$[\Lambda_1, \Lambda_2]_- = [\Lambda_a, A(p)]_- = [\Lambda_a, A^+(p)]_- = 0;\ \Lambda_a H(\lambda_1, \lambda_2) = \lambda_a H(\lambda_1, \lambda_2);$$

$$\lambda_a = 0, \pm 1; \pm 2; \ldots .$$

It can also be shown that there exists in $H(\lambda_1, \lambda_2)$ a unique vector $\phi_0(\lambda_1, \lambda_2)$ satisfying the equation $A(p)\phi_0(\lambda_1, \lambda_2) = 0$ for all $p \neq 0$. This means that the irreducible representations into which the representation (4) is decomposed are equivalent to the Fock representation of BCR. Using (2) – (5), we can readily verify that $[H_0 - \sum_q |q| A^+(q)A(q), A(p)] = 0$. Hence, by virtue of the irreducibility of $A(p)$ and $A^+(p)$ in each of the $H(\lambda_1, \lambda_2)$, there follows the possibility of representing H from (1) in the form

$$H = \frac{2\pi}{L}\sum_{q>0} q\{A^+(q)A(q) + A^+(-q)A(-q) + 2gV(q)[A^+(q)A^+(-q) +$$

$$+ A(q)A(-q)]\} + E(\lambda_1, \lambda_2),$$

where $E(\lambda_1, \lambda_2)$ is uniquely defined by the equation

$$H_0\phi_0(\lambda_1, \lambda_2) = E(\lambda_1, \lambda_2)\phi_0(\lambda_1, \lambda_2).$$

The problem of calculating the energy of the Fermi sustem thus reduces to the well studied Bose problem, from which the character of the spectrum of H, which we described above, follows in obvious fashion. Physically, the appearance of a continuous spectrum is connected with the fact that the system under consideration, being bounded in space, has nevertheless an infinite number of degrees of freedom, as is evidenced by the possibility of unlimited pair production: the momentum space may be discrete, but is not bounded.

The foregoing example shows that in describing finite Fermi systems (such as nuclei), limitations may arise on the interaction force, dictated by the requirement that the energy be positive. On the other hand, the same example points to the possible need for using non-Fock representations of commutation relations for the description of such systems.

[1] D. C. Mattis and E. H. Lieb, J. Math. Phys. 6, 304 (1965).

*To simplify the exposition, we assume that

$$\tilde{V}(p) = \frac{1}{2\pi}\int_{-L/2}^{L/2} dx\, V(x) \exp[ipx]$$

is a finite function, equal to zero when $p \geq p_{max}$ and that $V(0) = 0$. All that follows can be proved in the gneral case, too.

** By virtue of the periodic boundary conditions, the momentum variables p, q, etc. run through the values $\pm 2\pi n/L$, n = 0, 1, 2, ...

EMISSION STATISTICS OF A LASER WITH NONRESONANT FEEDBACK

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1. In [1,2] we proposed and investigated a laser with nonresonant feedback produced by radiation scattering. In this letter we report the results of a theoretical and experimental investigation of the statistical properties of the emission of such a laser. We shall show that the emission statistics of a laser with nonresonant feedback differs greatly from the emission statistics of ordinary lasers. The radiation intensity within extremely narrow solid angles is subject to strong fluctuations, and the distribution function of the intensity fluctuations coincides with the distribution function of the number of photons in one quantum state of black-body radiation at large occupation numbers.

2. Generation in a laser with nonresonant feedback is effected in a large number of low-Q modes that interact strongly with one another as a result of scattering. Their number L (for one polarization) is given by

$$L \simeq \Omega_{gen}/(\frac{\lambda}{D})^2, \tag{1}$$

where Ω_{gen}, D, and λ are the solid angle, diameter, and wavelength of the generated emission, respectively. Generation in a set L of scattering-coupled modes is due to the fact that the radiation losses of the set of interacting modes is much lower than the loss of any single mode [1].

The theoretical analysis of the emission statistics of a laser with nonresonant feedback is based on the following model: We consider an ensemble of a large number L of modes that interact nonlinearly with the active medium and interact linearly with one another via exchange of radiation (by scattering). The active medium is described by a set of M two-level atoms, and the radiation field in the i-th mode, by the number of photons n_i. A method similar to that developed in [3] yields a pilot equation for the probability $P_m^{n_1,n_2,\ldots n_L}$ of the state with n_i photons in the i-th mode (i = 1, 2, ..., L) and m atoms at the lower level, and also for the probability P_m^N of the state with $N = \sum_{i=1}^{L} n_i$ photons in all L modes and m atoms at the lower level.

In the stationary state, the pilot equation for the probability P_m^N determines fully the distribution function of the total number of photons, P^N, which turns out to be analogous, owing to saturation of the atoms, to the distribution function of the number of photons in one mode of an ordinary laser (it has a sharp maximum at $N = \bar{N}$ with relative width $\sim 1/\sqrt{N}$, [3-5]). From the pilot equation for the probability $P_m^{n_1,n_2,\ldots,n_L}$ in the stationary state we get the distribution function P^{n_i} of the number of photons in the i-th mode. Assuming that all $\bar{n}_i = \bar{n} >> 1$, this distribution takes a form

$$P^{n_i} = \frac{1}{\bar{n}} \exp(-\frac{n_i}{\bar{n}}) \quad (i = 1,2,\ldots,L) \tag{2}$$