

$$G^2 \sim 1 + \frac{\Omega^2}{4\Gamma^2 \ln(1/\Gamma\bar{\Gamma})}. \quad (12)$$

In (10) and (11), the second derivative is proportional to Γ^2 , i.e., it is inversely proportional to the square of the width of the dip. In the two-dimensional and three-dimensional cases, which are of practical importance, the second derivative equals $(2\Gamma^2 \ln 1/\Gamma\bar{\Gamma})^{-1}$ whereas the reciprocal of the half-width squared is $\bar{\Gamma}/\Gamma$. Their ratio is $(2\bar{\Gamma} \ln 1/\Gamma\bar{\Gamma})^{-1}$, so that the parameter determining the reliability of the frequency stabilization may be much smaller than the width of the dip and is determined, in practice, by the level width Γ .

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KINETIC EQUATION FOR A GAS OF SOLITONS

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Many problems describing the propagation of nonlinear waves in dispersive media reduce to the Korteweg - de Vries equation [1]

$$v_t + v_x + vv_x + v_{xxx} = 0 \quad (1)$$

The existing algorithm [2, 3] for the construction of solutions of (1) fail in practice in those cases when the initial profile $v(x, 0)$ has a complicated form and leads to the formation of a large number of solitary waves (solitons):

$$v = 3\alpha \operatorname{ch}^2 [\sqrt{\alpha}(x - ut)/2], \quad \alpha = u - 1 > 0, \quad (2)$$

where u is the wave velocity, and we assume that $\alpha \ll 1$. At the same time, such a situation is characteristic of a turbulent medium, and its investigation is of appreciable interest. We develop below a statistical method for the investigation of (1), leading to the possible use of the concept of a Boltzmann "gas" of solitons.

The Hamiltonian formalism for (1) is of the form

$$H = \frac{1}{2} \int dq (1 - q^2) v(q) v(-q) + \frac{1}{6} \int dq_1 dq_2 dq_3 v(q_1) v(q_2) v(q_3) \delta(q_1 + q_2 + q_3)$$

$$v = \int dq e^{iqx} v(q); \quad v(-q) = v^*(q); \quad \dot{v}(q) = iq \frac{\delta H}{\delta v(-q)}. \quad (3)$$

We consider a "gas" made up of a large number of solitons, the distance between which is on the average much larger than the characteristic width of the solitons $\alpha^{-1/2}$. We represent H in the form

$$Q = \sum_I H_i + V,$$

$$H_i = \frac{1}{2} \int dq_i (1 - q_i^2) v_i(q_i) v_i(-q_i) + \frac{1}{6} \int dq_i dq_i' dq_i'' v_i(q_i) v_i(q_i') v_i(q_i'') \times \delta(q_i + q_i' + q_i'');$$

$$V = \frac{1}{2} \sum_{i \neq j} \int dq_i dq_j (1 + q_i q_j) v_i(q_i) v_j(q_j) \delta(q_i + q_j) +$$

$$+ \frac{1}{6} \sum_{i \neq j \neq k} \int dq_i dq_j dq_k v_i(q_i) v_j(q_j) v_k(q_k) \delta(q_i + q_j + q_k).$$
(4)

The perturbation V for a rarefied soliton gas is connected with the overlap of the "tails" of the distributions (2), and is therefore small. This allows us to continue by using perturbation theory.

We transform V into an expression containing explicitly the dependence on only the distance R between the coordinates of the soliton centers. With the aid of (2) we get $v(q) = a(q) \exp(iqy)$, where the soliton-center coordinate is $y(t) = y_0 + at$. Substitution in (4) yields

$$V = \frac{1}{2} \sum_{i \neq j} V_{ij}(R_{ij}) + \frac{1}{6} \sum_{i \neq j \neq k} V_{ijk}(R_{ij}, R_{ik}),$$

$$V_{ij}(R) = \int dq V_{ij}(q) e^{iqR}; \quad R_{ij} = y_i - y_j,$$

$$V_{ij}(q) = V_{ij}^*(q) = V_{ij}(-q), \quad V_{ij}(q) = \Gamma_{ij} a_i(q) a_j^*(q);$$

$$V_{ijk} = \int dq_i dq_j dq_k V_{ijk}(q_i, q_j, q_k) \exp[i(q_i y_i + q_j y_j + q_k y_k)],$$

$$V_{ijk}(q_i, q_j, q_k) = \Gamma_{ijk} a_i(q_i) a_j(q_j) a_k(q_k) \delta(q_i + q_j + q_k),$$
(5)

where Γ_{ij} and Γ_{ijk} are factors cutting off the interaction at $\sqrt{\alpha}R < 1$. It is easy to see from (5) that

$$V_{ijk} = V_{ijk}(R_{ij}, R_{ik}) = V_{kij}(R_{ki}, R_{kj}) = V_{jki}(R_{jk}, R_{ji}).$$

Introducing canonical variables, the action (I) and the phase (y), with the aid of the relations $dH_i/dI_i = \alpha_i$ and $\dot{y}_i = \alpha_i$, we write down the Liouville equation for the distribution function $f(I_1, \dots; y_1, \dots; t)$

$$\frac{\partial f}{\partial t} + \sum_i \alpha_i \frac{\partial f}{\partial y_i} = \sum_i \left(\frac{\partial V}{\partial y_i} \frac{\partial f}{\partial I_i} - \frac{\partial V}{\partial I_i} \frac{\partial f}{\partial y_i} \right).$$
(6)

Using the smallness of the right side in (6) and averaging over the initial soliton coordinates y_0 , we arrive at the following kinetic equation for the multiparticle distribution function $F(I_1, I_2, \dots; t)$:

$$\frac{\partial F}{\partial t} = \pi \sum_{i \neq j \neq k} \int dq_i dq_j dq_k D_{ijk} \delta(a_i q_i + a_j q_j - a_k q_k) |V_{ijk}(q_i, q_j, q_k)|^2.$$

$$D_{ijk} \delta(q_i + q_j - q_k) F,$$

$$D_{ijk} = q_i \frac{\partial}{\partial l_i} + q_j \frac{\partial}{\partial l_j} - q_k \frac{\partial}{\partial l_k}.$$
(7)

Equation (7) does not contain two-soliton interactions, since for such interactions the conservation law connected with the appearance of the δ -function $\delta(q(\alpha_i - \alpha_j))$ can be satisfied only for identical solitons. Consequently, such processes have a small phase volume. This circumstance is a peculiarity of the one-dimensional character of the problem.

We now integrate (7) over all the actions, with the exception of a certain fixed one, and express, as usual, the three-soliton distribution functions in terms of single-soliton ones. This leads to an equation for the single-soliton function $\rho(l)$, which is the analog of the Boltzmann equation:

$$\frac{\partial \rho(l)}{\partial t} = \pi n^2 \int dl' dl'' \int dq dq' dq'' q \frac{\partial}{\partial l} \delta(qa + q'a' - q''a'') \times$$

$$\times \delta(q + q' - q'') |V(q, q', q'')|^2 \left(q \frac{\partial}{\partial l} + q' \frac{\partial}{\partial l'} - q'' \frac{\partial}{\partial l''} \right) \rho(l) \rho(l') \rho(l''),$$
(8)

where n is the density of the solitons. A stationary solution of (8), corresponding to the thermodynamic equilibrium of a soliton gas, is

$$\rho(l) = \rho_0 \exp(-H(l)/T_0) \approx \rho_0 \exp(-a^{3/2}/T),$$
(9)

where ρ_0 , T_0 , and T are constants, and the connection between H and α is obtained from (2) and (3).

The obtained solution (9) is remarkable if one turns to the known results for the energy spectrum of one-dimensional disordered systems [4, 5]. According to [2, 3], solitons with amplitudes $\alpha_i \sim |E_i|$, where E_i are discrete eigenvalues of the Schrodinger equation with potential $-v(x, 0)$, are formed from the initial state $v(x, 0)$. Therefore (9) leads to a spectrum $\rho(E) \sim \exp(-cE^{3/2})$, in accord with [4, 5]. An even more interesting connection with disordered system is obtained if one recalls that the effective potential of a disordered system, in which a state with large negative E is formed, has the form of the soliton (2) [6].

The method described above can be generalized to include conditions that make it necessary to consider a gas in which the role of the particle is assumed by a system of several correlated solitons.

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APPROXIMATE SOLUTION OF THE THREE-BODY PROBLEM WITH A LOCAL POTENTIAL

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As is well known, to solve the Faddeev equations it is necessary to know the behavior of the two-particle T-matrix off the mass shell. We shall point out one possibility of constructing such a T-matrix. Assume that we have a local short-range potential $V(r)$. The ℓ -th harmonic of the Fourier transform of this potential is given by

$$V_{\ell}(k, k') = \frac{1}{\pi^2} \int_0^{\infty} j_{\ell}(kr) j_{\ell}(k'r) V(r) r^2 dr \quad (1)$$

We shall approximate the local potential $V_{\ell}(k, k')$ by the aggregate of nonlocal potentials, using the Bateman method [1]. We obtain the following expression for the approximating potential $V_{\ell}(k, k')$:

$$\tilde{V}_{\ell}(k, k') = T_{\ell}[\theta(k, k') \mathcal{I}^{-1}]. \quad (2)$$

The solution of the Lippman-Schwinger equation with potential $V_{\ell}(k, k')$ is

$$T_{\ell}(k, k', z) = \text{Tr}[C(z) \theta(k, k')]$$

where

$$\begin{aligned} \theta_{ij}(k, k') &= V_{\ell}(k, s_j) V_{\ell}(k', s_i), \\ d_{ij} &= V_{\ell}(s_i, s_j), \\ I_{ij}(z) &= \int_0^{\infty} k^2 dk \frac{V_{\ell}(k, s_i) V_{\ell}(k, s_j)}{k^2 - \sqrt{2\mu_{12}} z - i\epsilon}, \end{aligned} \quad (3)$$

$$C_{ij}(z) = [(d + \mathfrak{g} \pi \mu_{12} I)^{-1}]_{ij},$$

s_i are parameters, $i, j = 1, \dots, n$, and μ_{12} is the reduced mass of the repelling particles.

It is seen from (2) that when k and k' equals one of the parameters s_i , the approximate potential $\tilde{V}_{\ell}(k, k')$ coincides with the local potential (1). It is clear that if the points s_i are uniformly distributed along the axes k and k' and if $n \rightarrow \infty$, the approximate potential $\tilde{V}_{\ell}(k, k')$ approaches the local potential (1). For most of the short-range potentials used in the calculations, we can confine ourselves to values of n that do not differ strongly from unity. This is possible as a result of the smoothness of the function $V_{\ell}(k, k')$ with respect to the variables k and k' . From the condition that the integral

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