mined by investigating not only the total pion spectrum, but also the Dalitz diagram. In particular, the characteristic effects can be observed by considering the distribution of the number of events along the resonance (along the line $M_{\pi\Lambda} = 1385$ MeV).

In conclusion, we wish to call attention to the rather single-valued character of (1). This is connected with the fact that the decay into the $Y_1(1385)$ and a pion occurs only in an s-wave and is described by one isotopic amplitude, owing to the quantum numbers of the $Y_0(1520)$ resonance ($T = 0$, $J^P = 3/2^-$) and the low kinetic energy of $Y_1(1385)$.

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ELEMENTARY SOLUTIONS OF THE QUANTUM PROBLEM OF THE MOTION OF A PARTICLE IN THE FIELD OF TWO COULOMB CENTERS

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The problem of the bound states of a particle in the field of two Coulomb centers has in some particular cases simple analytic solutions. These solutions can serve as a check on numerical calculations (similar calculations have been made recently in connection with mesic-molecule problems [1]). In addition, interest attaches to the very existence of such solutions for a problem that plays just as fundamental a role as the problem of the hydrogen atom in atomic physics. This problem has been under study for more than forty years, and it is surprising that these solutions were apparently not discovered earlier.

We denote the charges of the Coulomb centers by Z_1 and Z_2, and the distance between them by R. We shall assume that $Z_1 > 0$ and $Z_2 > Z_1$ and, in addition, that Z_1 and Z_2 are mutually prime integers. The common factor can be readily eliminated by a scale transformation. Then the variables separate in the ellipsoidal coordinates $\xi = (r_1 + r_2)/R$, $\eta = (r_1 - r_2)/R$, and φ. Then, representing the wave function in the form $\psi = F(\xi)G(\eta)\exp(im\varphi)$, we obtain for F an equation containing by way of a parameter only $Z_1 + Z_2$, and for G only $Z_1 - Z_2$. It follows therefore that if the energy of the system (without the interaction of the nuclei) is equal to $(Z_1 + Z_2)^2/2n_1^2$ ($n_1 = 1, 2, \ldots$) and the number of nodes of the function F in the interval $(1, +\infty)$ does not exceed $n_1 - |m| - 1$, then F corresponds to the one-center problem, i.e., it can be represented in the form

$$F = (\xi^2 - 1)^{m/2} P(\xi) \exp[-R(Z_1 + Z_2)\xi/2n_1], \qquad (1)$$

where P is a polynomial. The same holds true for the function G, if the energy is equal to $(Z_1 - Z_2)^2/2n_2^2$ and the number of nodes of G in the interval $(-1, +1)$ is not larger than $n_2 - |m| - 1$. If both conditions are simultaneously satisfied, then the two-center problem reduces to two hydrogenlike problems for the motion of a particle in the field of the charge $Z_1 + Z_2$ for the function F and of the charge $Z_1 - Z_2$ for the function G, and can thus be solved exactly.

The condition $E = (Z_1 + Z_2)^2/2n_1^2 = (Z_1 - Z_2)^2/2n_2^2$ can be satisfied for odd-odd Z_1 and Z_2 only if $n_1 = n(Z_1 + Z_2)/2$ and $n_2 = n(Z_1 - Z_2)/2$ $(n = 1, 2, \ldots)$. Then $E = -2/n^2$, i.e., the energy coincides with the energy levels of the He^+ system. For Z_1 and Z_2 having unlike parities we have $n_1 = n(Z_1 + Z_2)$, $n_2 = n(Z_1 - Z_2)$, and $E = -1/2n^2$, i.e., the energy should coincide with the energy levels of hydrogen. Satisfaction of these conditions still does not mean that the solution has an elementary form, since the aforementioned conditions imposed on the number of nodes may still not be satisfied.

If $Z_2 = 0$, then the equations for the functions F and G coincide exactly, and since both functions should be regular at the point $\xi = 1$, the functions themselves coincide, and the solution of the hydrogenlike problem takes the form

$$\psi = F(\xi)F(\eta)\exp(im\,\varphi). \qquad (2)$$

Thus, the hydrogenlike problem in ellipsoidal coordinates reduces to a single equation with two parameters - the energy and the separation constant - which must be chosen such as to make the solution regular at the three points -1, +1, and $+\infty$, i.e., we obtain, as it were, the double eigenvalue problem.

Inasmuch as the hydrogenlike functions F and G are regular at the points -1, +1, and $+\infty$, we can interchange ξ and η in the function (1), which is equivalent to the substitution $n_1 \rightleftharpoons n_2$ or else to a reversal of the sign of Z_2. We thus obtain the following reciprocity condition: if there exists an elementary solution for a system with charges Z_2 and Z_2, there exists for the same distance R and the same energy E an elementary solution for the system $Z_1 - Z_2$. The equations for the determination of R are:

(1) $R^3 + 2R^2 - 9R - 6 = 0,$
(2) $R^2 = x^2/6 + 17x/9 + 14/3;\ 15x^3 + 296x^2 + 1652x + 2688 = 0,$
(3) $23R^3 + 168R^2 - 576R - 720 = 0,$
(4) $R^2 = x^2/4 + x/2;\ 5x^3 + 38x^2 - 336x - 1120 = 0,$
(5) $5R^4 + 24R^3 - 71R^2 - 192R - 60 = 0.$

The problem of finding these solutions reduces to a search for values of R and of the separation constant such as to cause the vanishing of two determinants of order $n_1 - |m|$ and $n_2 - |m|$. An investigation of the behavior of the roots of these determinants at small and large values of R makes it possible to obtain the total number of elementary solutions for the given n_1, n_2, and m, to classify the states, etc. For example, when $m = 0$ the total number of solutions is $n_2[n_1 - n_2 + 1)/2] - 1$ ([ ] denotes the integer part of the number). Thus for $n_1 = 10$ and $n_2 = 5$ we have 14 solutions ($Z_1 = 3$, $Z_2 = 1$, $E = -2/25 = -0.08$). In some cases there can be for one term two or more points corresponding to elementary solutions. The table lists the simplest elementary solutions.

T a b l e

№	n_1, n_2	Z_1, Z_2	E	State $Z_2 > 0$	State $Z_2 < 0$	R	
I	3,2	5,1	-2	$3p\sigma$	$2p\sigma$	$\sqrt{10}/3$	
II	4,1	5,3	-2	$3s\sigma$	$2p\sigma$	$\sqrt{10}$	
III	4,1	5,3	-2	–	$2p\pi$	2.519	(1)
IV	4,2	3,1	-1/2	$4p\sigma$	$2p\sigma$	$\sqrt{3}$	
V	4,3	7,1	-2	$4p\sigma$	$3p\sigma$	0.726	(2)
VI	4,3	7,1	-2	$4d\sigma$	$3d\sigma$	2.241	(2)
VII	4,3	7,1	-2	$4d\pi$	$3p\pi$	$\sqrt{7}/3$	
VIII	5,1	3,2	-1/2	$4s\sigma$	$2p\sigma$	$\sqrt{15}$	
IX	5,1	3,2	-1/2	–	$2p\pi$	3.335	(3)
X	5,2	7,3	-2	$4s\sigma$	$3p\sigma$	3.911	(4)
XI	5,2	7,3	-2	$4p\sigma$	$3d\sigma$	5.330	(4)
XII	5,2	7,3	-2	$5p\sigma$	$2p\sigma$	0.735	(4)
XIII	5,2	7,3	-2	$4p\pi$	$3d\pi$	$\sqrt{21}$	
XIV	5,2	7,3	-2	–	$3p\pi$	3.313	(5)

We also write out the simplest wave function corresponding to solution I in the table ($Z_2 > 0$):

$$\psi = N(5\xi^2 - 2\sqrt{10}\,\xi - 7)(5\eta + \sqrt{10})\exp[-\sqrt{10}(\xi + \eta)/3]. \tag{3}$$

Solutions I and VI can be compared with the calculations in [1] (Figs. 3 and 5). Within the limits of the accuracy of the drawing, the points corresponding to these solutions lie on the calculated curves that are presented there. Knowing the wave functions, it is easy to calculate such quantities as the slopes of the terms (from the virial theorem), the polarizability, etc. for these solutions.

Besides the solutions of type (2), there exist when $m \neq 0$ also elementary solutions of the type

$$F(\xi) = [(\xi - 1)/(\xi + 1)]^{m/2} P(\xi)\exp[-(Z_1 + Z_2)\,\xi R/2n_1], \tag{4}$$

which diverge when $\xi = -1$. The reciprocity condition does not hold for these solutions, and they exist only when $Z_2 < 0$. In our table, such solutions are III, IX, and XIV.

Both types of polynomial solutions were considered in [2] for each of the equations for F and G separately. The author has, however, drawn the incorrect conclusion that the remaining solutions are irregular, and arrived at meaningless results.

We note, finally, that for the cases $Z_1 = \pm Z_2$, there are no elementary solutions, only the function $F(Z_1 = Z_2)$ or the function $G(Z_1 = -Z_2)$ can have an elementary form here.

In conclusion, I am grateful to I. V. Komarov. The idea of this paper arose as a result of a discussion held with him regarding this group of problems.

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SYMMETRY OF RELATIVISTIC PROBLEMS

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The advantages ensuing from the use of higher symmetries was demonstrated in a recent paper [1] with the nonrelativistic Coulomb problem as an example. The discrete spectrum of this problem can be related either with the irreducible infinite-dimensional representation of the group O(4,1) [2] or, more interestingly (see [1]), with the representation of the group O(4,2) [3]. To understand the connection between the internal symmetries and the Lorentz group, it is of interest to study the simplest relativistic models [3]. In the present paper we show that a free Dirac particle has additional integrals of motion, which form together with the angular momentum the SL(2,C) group of SU(2) symmetry. We show that the wave functions belonging to one energy level realize a not-fully-reducible infinite-dimensional representation of the SL(2,C) group. An additional integral of motion was found earlier [5] in the relativistic Coulomb problem, and its physical meaning was explained by Biedenharn [6]. In the present article we construct one more integral of motion, which forms together with the available ones a group of SU(2) symmetry. The existence of this group explains well the double degeneracy of the relativistic Coulomb problem. The Hamiltonian for the free Dirac equation

$$(-\rho_2 \vec{\sigma}\, \vec{\nabla} + \rho_3 E - m)\psi = 0; \quad (\hbar = c = 1) \tag{1}$$

commutes with the angular momentum j and with the Dirac operator $K = \rho_3(\vec{\sigma}\cdot\vec{L} + 1)$. For states with fixed energy there is an infinite set of wave functions with $j = 1/2, 3/2, 5/2, \ldots$ and $k = \pm|j + 1/2|$ (where k is the eigenvalue of the Dirac operator), forming a basis of space H. It is easy to verify that the operators

$$X_1 = \frac{\vec{\sigma} p}{|p|}; \; X_2 = \frac{\rho_3 \vec{\sigma} A}{|k|}; \; A = \frac{Lp - pL}{2|p|}; X_3 = \frac{K}{|k|} \tag{2}$$

commute with the angular momentum $\vec{j}$ and with the free Hamiltonian, and satisfy the relations

$$[X_i X_k] = 2i\epsilon_{ik\ell} X_\ell; \; [X_i X_k]_+ = 2\delta_{ik}; \; (i, k = 1 \ldots 3). \tag{3}$$

In two-dimensional space of states with fixed energy, $\vec{j}^2$ and j_z, these operators are ordinary Pauli σ_i matrices. We define the three operators

$$L' = \frac{1}{2}[2L + \vec{\sigma} - p\frac{\vec{\sigma}p}{p^2} + \rho_3(\vec{\sigma} - p\frac{\vec{\sigma}p}{p^2})];$$