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SYMMETRY OF RELATIVISTIC PROBLEMS

I. A. Malkin and V. I. Man'ko
P. N. Lebedev Physics Institute, USSR Academy of Sciences
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The advantages ensuing from the use of higher symmetries was demonstrated in a recent paper [1] with the nonrelativistic Coulomb problem as an example. The discrete spectrum of this problem can be related either with the irreducible infinite-dimensional representation of the group O(4,1) [2] or, more interestingly (see [1]), with the representation of the group O(4,2) [3]. To understand the connection between the internal symmetries and the Lorentz group, it is of interest to study the simplest relativistic models [3]. In the present paper we show that a free Dirac particle has additional integrals of motion, which form together with the angular momentum the SL(2,C) group of SU(2) symmetry. We show that the wave functions belonging to one energy level realize a not-fully-reducible infinite-dimensional representation of the SL(2,C) group. An additional integral of motion was found earlier [5] in the relativistic Coulomb problem, and its physical meaning was explained by Biedenharn [6]. In the present article we construct one more integral of motion, which forms together with the available ones a group of SU(2) symmetry. The existence of this group explains well the double degeneracy of the relativistic Coulomb problem. The Hamiltonian for the free Dirac equation

$$(-\rho_2 \vec{\sigma}\vec{\nabla} + \rho_3 E - m)\psi = 0; \quad (\hbar = c = 1) \qquad (1)$$

commutes with the angular momentum j and with the Dirac operator $K = \rho_3(\vec{\sigma}\cdot\vec{L} + 1)$. For states with fixed energy there is an infinite set of wave functions with $j = 1/2, 3/2, 5/2, \ldots$ and $k = \pm|j + 1/2|$ (where k is the eigenvalue of the Dirac operator), forming a basis of space H. It is easy to verify that the operators

$$X_1 = \frac{\vec{\sigma} p}{|p|}; \quad X_2 = \frac{\rho_3 \vec{\sigma} A}{|k|}; \quad A = \frac{Lp - pL}{2|p|}; \quad X_3 = \frac{K}{|k|} \qquad (2)$$

commute with the angular momentum $\vec{j}$ and with the free Hamiltonian, and satisfy the relations

$$[X_i X_k] = 2i\epsilon_{ik\ell} X_\ell; \quad [X_i X_k]_+ = 2\delta_{ik}; \quad (i, k = 1 \ldots 3). \qquad (3)$$

In two-dimensional space of states with fixed energy, $\vec{j}^2$ and j_z, these operators are ordinary Pauli σ_i matrices. We define the three operators

$$L' = \frac{1}{2}\left[2L + \vec{\sigma} - p\frac{\vec{\sigma} p}{p^2} + \rho_3\left(\vec{\sigma} - p\frac{\vec{\sigma} p}{p^2}\right)\right];$$

$$\Sigma' = -\rho_3 \vec{\sigma} + p \frac{\vec{\sigma} p}{|p|^2} + \rho_3 p \frac{\vec{\partial} p}{p^2} ;$$

$$A' = \frac{1}{2|p|} [pL' - L'p]. \tag{4}$$

They commute with the Hamiltonian and satisfy the following commutation relations:

$$L'L' = -A'A' = iL'; L'A' + A'L' = 2iA';$$
$$[L'_i \Sigma'_k = [A'_i \Sigma'_k] = 0 \quad \Sigma'\Sigma' = 2i\Sigma'. \tag{5}$$

The operators (4) form the SL(2,C) group of SU(2) symmetry. (The Casimir operators are in this case $(\vec{L}' + i\vec{A}')^2 = (\vec{L}' - i\vec{A}')^2 = -1$, $(\Sigma') = 3$).

Let us consider the two operators

$$J = L' + \frac{1}{2} \vec{\Sigma}'; K = A' + \frac{i}{2} \vec{\Sigma}', \tag{6}$$

obeying the commutation relations of the SL(2,C) group

$$J J = -KK = iJ ; \; J K + K J = 2iK . \tag{7}$$

The operator $\vec{J}$ is the angular-momentum operator. The two Casimir operators for this group

$$j_1 = (J + iK)^2 ; \; j_2 = (J - iK)^2 \tag{8}$$

determine the representation of this SL(2,C) group [7].

A simple calculation yields

$$j_1 = -1; \; j_2 = -2(1 + X_1) K; \; j_2^2 = 0. \tag{9}$$

This means that this representation is not fully reducible. In the aforementioned two-dimensional space the operator j_2 can be reduced only to triangular form.

Representations of the SL(2,C) group of this type were first investigated by D. P. Zhelobenko [8]. The existence of these representations in the free Dirac equation is due to the relativistically invariant character of this equation, as is the independence of the parameters j_1 and j_2 of the energy. It is probable that the difficulties in the relativization of the SU(6) symmetry group, and also in the relativistic treatment of the internal symmetries, may be connected with the fact that only relativistic representations have been used all the time. The simple example considered here shows that for relativistic invariance it is also necessary to use not-fully-reducible representations. From the formal point of view, the appearance of a not-fully-reducible representation is due to the fact that the metric in H space is not positive definite ($\vec{u}^+ \gamma^0 \vec{u} = 2m$). The states of the Dirac particles in the Coulomb field are fully analogous to the states of the free particle [6], and consequently have the SL(2,C) symmetry of the SU(2) symmetry. In a Coulomb field there also exists a group of SU(2) symmetry of somewhat modified operators (2).

The operator

$$X_2' = [K^2(H^2 - m^2) + \ell^4 Z^2 m^2]^{-1/2} [-iK\rho_1(H - m\rho_3) - \ell^2 mZ \frac{\vec{\sigma}\mathbf{r}}{r}] \quad (10)$$

is an integral of motion in the relativistic Coulomb problem [5].

The operator

$$X_1' = [(H^2 - m^2)K^4 + \ell^4 Z^2 m^2 K^2]^{-1/2} [K^2\rho_1(H - m\rho_3) - i\ell^2 mZ \frac{K\vec{\sigma}\mathbf{r}}{r}] \quad (11)$$

is also an integral of the motion.

The operators $X_1'X_2'X_3'$ obey the commutation relations (3). The existence of a symmetry group of SU(2) symmetry in the relativistic Coulomb problem explains the double degeneracy with respect to the number j. The details of the calculation will be published later.

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UNITARY STRUCTURE OF THE PARITY- AND STRANGENESS-CHANGING NONLEPTONIC WEAK INTERACTION

E. M. Lipmanov
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It is shown in this note that in the model of the unified electromagnetically-weak interaction [1], the assumption that the (V + A)-form strangeness-changing hadron currents in the nonleptonic current-current, $|\Delta S| = 1$, CP-odd weak interaction makes it possible to determine the unitary structure of the latter, which is in satisfactory quantitative agreement with the experimental data on $K \to 2\pi$ decays and in general qualitative correspondence with the nonleptonic rule $|\Delta T| = 1/2$ for all nonleptonic processes.

The assumption that in the nonleptonic current-current interaction the strangeness-changing currents have a Lorentz (V + A) form, and the strangeness conserving currents have a (V - A) form, lifts, as is well known, the forbiddenness [3] of the $K_1^0 \to 2\pi$ decays in the approximation of exact unitary symmetry of strong interactions. Accepting this assumption and supposing that the hadron currents of the unified electromagnetically-weak interaction [1] transform like the components of a unitary octet, we obtain with the aid of the tables of the C-G coefficients of the SU(3) symmetry group [4] the following unitary structure of