Here $\{\ldots\}_{p_x x}$ are the classical Poisson brackets, $\mathcal{E}$ the electric field intensity, and f_0 the Fermi function. We present without calculations the expression for $j(x)$ obtained for the configuration of Fig. 1 under the assumption that $W \approx 1$ (but $(1 - W)\kappa L \gg R$), when the band spectrum (1) represents slightly broadened Landau levels corresponding to the trajectories shown dashed in Fig. 1:

$$j_{cl}(x) = \frac{e^4 \mathcal{E}}{(2\pi\hbar)^3} \frac{4(1-W)}{f_0} \left(\frac{eH\hbar}{c}\right)^2 \left(\frac{H}{|\nabla H|}\right)^2 \int dp_z \frac{\partial S}{\partial \zeta} / S^2(\zeta, p_z).$$

The integration is over all the p_z for which magnetic breakdown takes place, and $S(\zeta p_z)$ is the area enclosed by the dashed trajectory in Fig. 1 (ζ is the Fermi energy).

The inequalities $R \ll \kappa L \ll \ell$, which are necessary for the occurrence of the "traps," can be readily satisfied in fields of 10^5 Oe if L lies between 1 and 10 cm. When $R \gtrsim \kappa L$, the tunnel transitions between the "traps" become appreciable, and the character of motion is greatly changed. These questions will be dealt with in a separate article.

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EXACT SOLUTION OF THE DECORATED ISING MODEL WITH NONMAGNETIC IMPURITIES

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The free energy, and hence all the remaining thermodynamic quantities, are known at present for only one system that undergoes a phase transition and has a finite interaction radius, namely the two-dimensional Ising lattice with nearest-neighbor interaction. This problem was solved by Onsager [1]. Great interest attaches therefore to an investigation of the phase transition in other models that admit of an exact solution. We obtain in this paper an exact solution of the decorated plane Ising model with nonmagnetic impurities. We define as decorated the lattice shown in Fig. 1 and consisting of two sublattices. The light and dark circles represent different atoms, A and B respectively, with magnetic moments that can be directed either upward or downward ($\sigma = \pm 1$). Only the neighbors joined by solid lines interact. The Hamiltonian of this system has the usual form for the Ising model[1]:

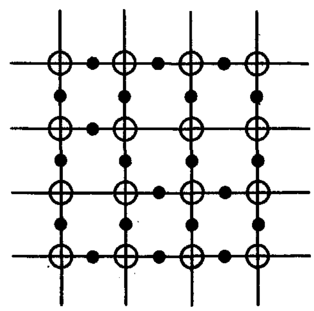

Fig. 1

[1] A similar model (without impurities) was investigated in detail by Syozi and Nakano [2].

$$H = -I \sum_{ij}{}' \sigma_i \sigma_j, \tag{1}$$

where the prime denotes that the summation is carried out only over the pairs of interacting neighbors. Recognizing that $\sigma = \pm 1$, we eliminate in the usual manner σ from the exponent in the partition function (see, e.g., the book of Landau and Lifshitz [3]), and sum over the spin configurations of the B-atoms:

$$Z = \sum_{\{\sigma\}} \exp(I/T \sum_{ij}{}' \sigma_i \sigma_j) = (\mathrm{ch}\, I/T)^{4N} \sum_{\{\sigma\}} \prod_{ij}{}'(1 + \sigma_i \sigma_j \mathrm{th}\, I/T) =$$
$$= (\mathrm{ch}\, I/T)^{4N} 2^{2N} \sum_{\{\sigma^A\}} \prod_{ij}{}' (1 + \sigma_i^A \sigma_j^A \mathrm{th}^2 I/T). \tag{2}$$

Here σ^A - spins of the A-atoms, N - number of unit cells, and

$$1/2^N \sum_{\{\sigma A\}} \prod_{ij}{}' (1 + x \sigma_i^A \sigma_j^A) = S(x) = \exp[Nf(x)], \tag{3}$$

where in the limit of very large N [1, 3]

$$f(x) = 1/8\pi^2 \iint_0^{2\pi} [(1 + x^2)^2 - 2x(1 - x^2)(\cos\omega_1 + \cos\omega_2)] d\omega_1 d\omega_2. \tag{4}$$

Thus, the difference between the decorated Ising lattice and the ordinary one is trivial, namely, here we have

$$x = \mathrm{th}^2(I/T). \tag{5}$$

Assume now that M out of the total number 2N of the B-atoms are replaced by nonmagnetic atoms, which are randomly distributed and rigidly fixed. Physically this means that the impurity atoms have been introduced into the sample at a sufficiently high temperature, and the time of relaxation of their position greatly exceeds the spin relaxation time. Then the partition function of the model with the impurities is

$$Z^* = (\mathrm{ch}\, I/T)^{4N-2M} 2^{3N-M} S^*(x), \tag{6}$$

where

$$S^*(x) = 2^{-N} \sum_{\{\sigma^A\}} \prod_{ij}{}' (1 + \tilde{x} \sigma_i^A \sigma_j^A). \tag{7}$$

Here $\tilde{x} = x$ or $\tilde{x} = 0$, depending on whether the bond joining the A-atoms contains the usual B-atom or the nonmagnetic impurity atom.

It is easy to see that $S^*(x)$ can be expressed in terms of the M-th derivative of $S(x)$:

$$S^*(x) = \frac{x^{2N-M}}{M!} \left. \frac{d^M S(z)}{dz^M} \right|_{z=1/x}. \tag{8}$$

Strictly speaking, the value of $S^*(x)$ obtained by differentiation corresponds to a partition function averaged over all possible impurity distributions. However, in the thermodynamic limit of interest to us ($N \to \infty$) the statistical weight of the nonrandom distributions vanishes.

Using (8), we express the derivative in terms of a contour integral:

$$S^*(x) = \frac{x^{2N-M}}{2\pi i} \oint \exp\{ N[f(z) - \frac{M+1}{N} \ln (z - \frac{1}{x})]\} dz, \tag{9}$$

where the contour surrounds the point $1/x$. We note that although the function $f(z)$ is not regular, $\exp[Nf(z)]$ is analytic in the entire complex plane for all finite N. We can thus deform the integration contour and use the saddle-point method. The asymptotic value of $S^*(x)$ as $N \to \infty$ is determined by the value of the integrand at the saddle point:

$$S^*(x) \sim x^{2N(1-c)} \exp\{N[f(z_0) - 2c\ln(z_0 - \frac{1}{x})]\}. \tag{10}$$

Here $c = M/2N$ is the impurity concentration, and the position of the saddle point as a function of x is given by the equation

$$f'(z_0) = \frac{2c}{z_0 - 1/x}. \tag{11}$$

The derivative $f'(z)$ which enters in this equation can be expressed in terms of the complete elliptic integral of the first kind $K(\kappa)$:

$$f'(z) = \frac{2z}{1+z^2} - \frac{1}{2}\left(\frac{2}{\pi}K(\kappa) - 1\right)\frac{d\ln\kappa}{dz}, \tag{12}$$

where

$$\kappa = \frac{4z(1-z^2)}{(1+z^2)^2}; \quad \frac{d\ln\kappa}{dz} = \frac{1}{z} - \frac{2z}{1-z^2} - \frac{4z}{1+z^2}. \tag{13}$$

We now obtain from (6) and (8) the free energy of the considered model with nonmagnetic impurities:

$$-F/NT = 4(1-c)\ln \operatorname{sh} I/T + (3-2c)\ln 2 + f(z_0) - 2c\ln(z_0 - 1/x). \tag{14}$$

From this we easily get, differentiating with respect to the temperature, the expressions for the internal energy and for the specific heat.

The phase transition occurs when the saddle point coincides with a singular point of the function $f(z)$, the singularities of which lie on circles of radius $\sqrt{2}$ with centers at the points $z = \pm 1$. Since $0 < x < 1$, the saddle point can coincide only with the singular point $z_c = \sqrt{2} + 1$, where $f'(z_c) = 1\ \sqrt{2}$. Using (11), we obtain an equation for the transition temperature:

$$x_c = \operatorname{th}^2 I/T_c = [1 + \sqrt{2}(1-2c)]^{-1}. \tag{16}$$

In the limiting cases we have

$$\frac{I}{T_c} = \begin{cases} \operatorname{arcth}(\sqrt{1+\sqrt{2}} + 1) + c/\sqrt{1+\sqrt{2}} \approx 0{,}764 + 0{,}664c, & c \ll 1 \\ \frac{1}{2}\ln\frac{2\sqrt{2}}{1-2c} & 0 < \frac{1}{2} - c \ll 1. \end{cases} \tag{17}$$

When $c > 1/2$, there is no phase transition, since our system breaks up at such impurity concentrations into non-interacting subsystems of finite dimensions.

Using the expansion of $f'(z)$ near $z_c = \sqrt{2} + 1$,

$$f'(z) = \frac{1}{\sqrt{2}} + \frac{z - z_c}{(1+\sqrt{2})^2}\left[+3 + \sqrt{2} + \frac{2}{\pi}\ln\frac{2\sqrt{2}(\sqrt{2}+1)}{|z - z_c|}\right], \quad |z - z_c| \ll 1, \tag{18}$$

we obtain for the specific heat near the critical point an expression that is valid when $I|\tau|T_c = I|T - T_c|/T_c^2 \ll 1$:

$$\frac{C}{N} = 4\frac{I}{T_c^2}(1-2c)\left\{\frac{\sqrt{2}+1-4c}{\sqrt{2}} + \frac{(1-2c)[1+\sqrt{2}(1-2c)]}{2c} \times \right. \times\left(1 - \frac{(1+\sqrt{2})^2\pi}{8c}\Lambda'(\alpha|\tau|)\right). \tag{19}$$

We have introduced here the function $\Lambda(\xi)$, defined as the solution of the equation

$$\eta \ln \eta = -\xi \tag{20}$$

for small ξ and η:

$$\eta = \Lambda(\xi). \tag{21}$$

Its derivative are

$$\Lambda'(\xi) = \Lambda(\xi)/(\xi - \Lambda(\xi)); \; \Lambda''(\xi) = \Lambda'^3(\xi)/\Lambda(\xi). \tag{22}$$

Plots of all the functions are shown in Fig. 2. The coefficient $\alpha(c)$ in (19) equals

$$\alpha(c) = \frac{(1+\sqrt{2})\pi(1-2c)}{8c\,x_c^{1/4}}\frac{I}{T_c}\exp\left\{-\frac{(1+\sqrt{2})^2\pi}{8c} \times \left(1 - 2c\frac{3+\sqrt{2}}{(1+\sqrt{2})^2\pi}\right)\right\}.$$

Thus, unlike Onsager's well known result [1], the specific heat of the model with nonmagnetic impurities remains finite at the transition point at nonzero impurity concentrations, and its order of magnitude is $1/c$. When $|\tau| \to 0$, the first derivative of the specific heat with respect to the temperature diverges like Λ''. When $c \ll 1$, the specific heat behaves in the vicinity of $\ln|\tau| \ll 1/c$ like $\ln|\tau|$.

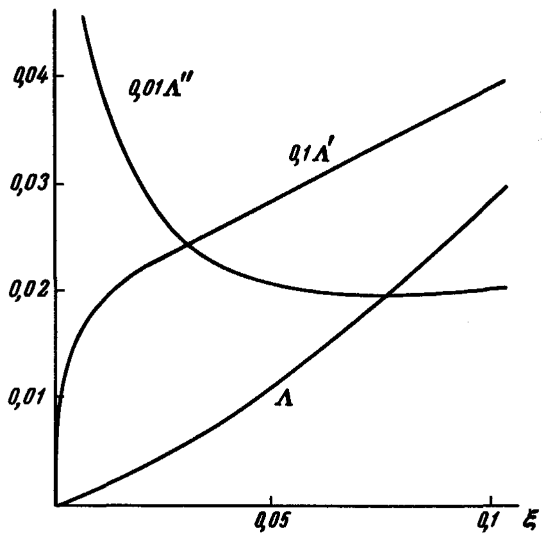

Fig. 2

The correlation function of the A-atoms of the model with impurities can be obtained in a similar manner:

$$\langle \sigma_l^A \sigma_j^A \rangle \equiv g^A(x, \vec{R}) = g_0(z_0, \vec{R}); \quad \vec{R} = \vec{R}_l - \vec{R}_j, \tag{24}$$

where $g_0(z, \vec{R})$ is the correlation function of the usual quadratic Ising lattice, and z_0 is determined from (11). Using the well known results of Kaufman and Onsager [4], Yang [5], and Fisher [6] for the function g_0, we obtain the following:

a) At the Curie point, the correlation function decreases in the usual manner like $R^{-1/4}$.

b) The correlation radius increases near T_c like $[\beta(c)\Lambda(\alpha\tau)]^{-1}$.

c) The spontaneous magnetization below the transition point behaves like $[\beta(c)\Lambda(\alpha\tau)]^{1/8}$. Accurate to a factor on the order of unity, we have

$$\beta(c) = \exp\frac{(1+\sqrt{2})^2\pi}{8c}\left(1 - 2c\frac{3+\sqrt{2}}{\pi(1+\sqrt{2})^2}\right) \tag{25}$$

We note that the lattice magnetization due to the A-spins does not reach saturation at T = 0, since a fraction of the A-atoms is isolated from the remainder of the system as a result of the nonmagnetic impurities.

In conclusion, we can express the hopes that the character of the singularities obtained in this paper, as well as the (qualitative) dependence of the transition temperature on the impurity concentration, are possessed not only by the model under consideration, but also by a broader class of systems, although the symmetrical form of the specific-heat singularity is a typical property of this model, connected with the Kramers-Wannier symmetry of the initial impurity-free model.

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POSSIBILITY OF OBSERVING ZERO SOUND IN NUCLEI BY RADIATIVE PION CAPTURE

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It is well known [1, 2] that there are four possible types of volume collective excitations of nulcear matter (NM), which can be regarded, in accord with the Landau theory of the Fermi liquid, as different types of zero sound in NM: density waves, spin waves, isospin waves, and coupled spin-isospin waves (henceforth denoted by the indices 0, s, i, and si). In this paper we determine the information that can be obtained concerning these excitations by investigating the interaction between slow pions and nuclei.

The exceptional properties of slow pions (with energy up to several MeV) are connected with their small scattering length compared with the radius of action of the nuclear forces and with the average distance between the nucleons of the nucleus (Ericson [3]), as a result of which the interaction between a slow pion and a nucleus can be described (just like the interaction between a slow neutron and a molecule or a crystal) with the aid of a sum of Fermi pseudopotentials. If we confine ourselves to a consideration of the interaction in the s-state, then the effective Hamiltonians of the π^--meson radiative capture and of the scattering of the pion by the nucleus are [4, 5]

$$H^{(r)} = 4\pi i \sum_{\ell} A_0 \tau^-(\vec{\epsilon}\vec{\sigma}_\ell)\delta(r - r_\ell), \tag{1}$$

$$H^{(s)} = 4\pi \sum_{\ell}(B_0 + B_1\vec{\tau}_\pi\vec{\tau}_\ell)\delta(r - r_\ell), \tag{2}$$