Pis'ma v ZhETF, vol. 86, iss. 3, pp. 179 – 182

Thermodynamic quantities for gases in static spherically symmetric backgrounds possessing a horizon

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Submitted 24 April 2007
Resubmitted 14 June 2007

We use the brick-wall method to study the thermodynamic quantities for the perfect relativistic gases in a generic spherically symmetric and static background spacetimes possessing a horizon. We employ the Wentzel-Kramers-Brillouin (WKB) approximation on the Teukolsky-type master equation. We show that the entropy density, energy density, pressure and state equation all contain a subleading term, which depends on the spins of the particles of gases. When particularizing to Schwarzschild and Reissner-Nördstrom geometries we recover the results previously found in the literature for those geometries.

PACS: 04.70.Dy

1. Introduction. Logarithmic corrections to Bekenstein-Hawking entropy have been extensively studied in the last several years and the existing computations of logarithmic corrections have very different physical starting points, such as the Cardy formula [1], and thermal fluctuations [2]. In particular, logarithmic corrections to Bekenstein-Hawking entropy due to spin fields propagating in black hole background were obtained in the literature [3 – 6] by the brick-wall method [7] and were found to be spin-dependent. Hawking radiation via tunneling from black hole has also been studied widely by using the Parikh-Wilczek method [8 – 13] and the same results, i.e. Hawking radiation is not exactly thermal radiation, information is preserved in black hole formation and evaporation, are obtained, as argued by Hawking in Ref. [14].

One generally assumes that the thermodynamic quantities in curved spacetime take the same forms as that in flat spacetime. Obviously this assumption is inconsistent with the results mentioned above. In fact, Li [15, 16] applied the quantization procedure referred to as Boulware vacuum state [17] and Killing time t to study the thermodynamic quantities for a perfect relativistic gas around the Schwarzschild black hole and the Reissner-Nordstrom black hole and found that near the event horizon there exist spin-dependent terms beyond the expected Minkowskian high-temperature contribution. In this paper, we extend this study to general spherically symmetric and static background spacetimes. We investigate the effects of spins on the thermodynamic quantities for the perfect relativistic gases with spin $s = 1/2$, 1, 3/2 and 2 by using the brick wall method. Some doubts on the validity of the brick-wall method are expressed [18, 19], but Mukohyama and Israel [20] have shown that these objections to be overcome when the ground state is correctly identified and that the local description of the statistical mechanics is equivalent to that of the quantized field in the curved background, which is defined globally and whose ground state is the Boulware state.

2. Space-time. The metric of a static spherically symmetric geometry can be written as

$$ds^2 = e^{2U(r)}dt^2 - e^{-2U(r)}dr^2 - R^2(r)(d\theta^2 + \sin^2\theta d\varphi^2). \tag{1}$$

We establish the following null tetrad vectors

$$\begin{aligned} l^\mu &= e^{-2U}\delta_0^\mu + \delta_1^\mu, \\ n^\mu &= \frac{1}{2}\delta_0^\mu - \frac{1}{2}e^{2U}\delta_1^\mu, \\ m^\mu &= \frac{1}{\sqrt{2}R}\delta_2^\mu + \frac{i}{\sqrt{2}R\sin\theta}\delta_3^\mu. \end{aligned} \tag{2}$$

Then we use the Newman-Penrose formula [21] to get the non-vanishing spin coefficients and component of the Weyl tensor

$$\begin{aligned} \alpha = -\beta &= -\frac{\operatorname{ctg}\theta}{2\sqrt{2}R}, \quad \rho = -\frac{R'}{R}, \\ \mu &= -\frac{R'}{2R}e^{2U}, \quad \gamma = \frac{1}{2}U'e^{2U}, \\ \Psi_2 &= -\frac{R''}{2R}e^{2U} - \frac{R'}{R}U'e^{2U}. \end{aligned} \tag{3}$$

Equations (3) show that the metric (1) is of Petrov type-D.

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3. Thermodynamic quantities. In type-D space-time, using the result of Teukolsky [22], the field equations of the Weyl neutrino ($s = 1/2$), electromagnetic ($s = 1$), mass-less Rarita-Schwinger ($s = 3/2$) and gravitational ($s = 2$) fields for the course free case can be combined into [3]

$$\{[D-(2s+1)\rho](\Delta-2s\gamma+\mu)-[\delta+(2s-2)\alpha]\times \times(\bar{\delta}-2s\alpha)-(2s-1)(s-1)\Psi_2\}\Phi_{+s}=0,$$
$$\{[\Delta+(2s-2)\gamma+(2s+1)\mu](D-\rho)-[\bar{\delta}+(2s-2)\alpha]\times \times(\delta-2s\alpha)-(2s-1)(s-1)\Psi_2\}\Phi_{-s}=0, \tag{4}$$

where $D = l^\mu\partial_\mu$, $\Delta = n^\mu\partial_\mu$, $\delta = m^\mu\partial_\mu$ are the directional derivatives. The first equation in Eq. (4) is for spin states $p = s$ and the second one for $p = -s$. In the quantization procedure referring to Boulware vacuum state and Killing time t, the mode functions Φ_p of spin fields around the static black hole are given by

$$\Phi_p = r^{p-s}\cdot {}_pR_{lE}(r)\cdot {}_pY_l^m(\theta,\varphi)\cdot e^{-iEt}, \tag{5}$$

where l, m are integers satisfying the inequalities $l \geq s$ and $-l \leq m \leq l$, ${}_pY_l^m(\theta,\varphi)$ is a spin-weighted spherical harmonic and ${}_pR_{lE}(r)$ satisfies the Teukolsky-type master equation:

$$\left\{\frac{d^2}{dr^2}+\left(2(1-p)U'e^{2U}+2(p+1)e^{2U}\frac{R'}{R}\right)\frac{d}{dr}+ +\left(\frac{E^2}{e^{4U}}+B(r)+iEC(r)-\frac{(l-p)(l+p+1)}{R^2e^{2U}}\right)\right\}{}_pR_{lE}(r)=0, \tag{6}$$

where

$$B(r) = -(2p^2-3p+2)\frac{R''}{R}+2(p+1)\frac{R'^2}{R^2}- -2p(U''+2U'^2)+2p(2p+1)U'\frac{R'}{R}, \tag{7}$$
$$C(r) = 2pe^{-2U}\frac{R'}{R}.$$

We use the Wentzel-Kramers-Brillouin (WKB) approximation and write

$${}_pR_{lE}(r) = \exp\left[iS(r,p,l,E)\right], \tag{8}$$

Then we obtain the radial wave number by using the formula $k(r,p,l,E) \equiv \partial_r S(r,p,l,E)$

$$k^2 = \frac{E^2}{e^{4U}}+B(r)-\frac{(l-p)(l+p+1)}{R^2e^{2U}}. \tag{9}$$

According to the semi-classical quantization rule, the wave number is quantized as

$$\int_{r_H+\varepsilon}^{r_H+\varepsilon+h} kdr = n\pi, \quad n\in N, \tag{10}$$

where r_H denotes the event horizon of the black hole, ε is the distance of the brick wall from the horizon and h is the thickness of the brick wall, which satisfy the inequality $0 < \varepsilon \ll h \ll r_H$. Then the number of eigen-states with energy smaller than E is given by

$$g(E) = \sum_p\sum_l(2l+1)n = = \frac{1}{\pi}\sum_p\int_s^{l_{max}}(2l+1)dl\int_{r_H+\varepsilon}^{r_H+\varepsilon+h}kdr = =\frac{2}{3\pi}\sum_p\int_{r_H+\varepsilon}^{r_H+\varepsilon+h}\frac{R^2}{e^{4U}}\left[E^2+e^{4U}B(r)-(s-p)\frac{e^{2U}}{R^2}\right]^{3/2}dr, \tag{11}$$

where the upper limit l_{max} is determined by Eq. (9).

The free energy of the system at the inverse Hawking temperature is given by

$$-\beta F = \pm\sum_\alpha \ln\left(1\pm e^{-\beta E_\alpha}\right), \tag{12}$$

where the plus sign corresponds to the Fermi case and the minus sign to the Bose case. We use Eq. (11) to determine the state density and obtain the free energy

$$F = \mp\frac{1}{\beta}\int_0^\infty dE\frac{dg(E)}{dE}\ln(1\pm e^{-\beta E}) = = -\frac{15+(-1)^{2s}}{360}\frac{\omega\pi^3}{\beta^4}\int_{r_H+\varepsilon}^{r_H+\varepsilon+h}\frac{R^2}{e^{4U}}dr- -\frac{3+(-1)^{2s}}{24}\frac{\pi}{\beta^2}\int_{r_H+\varepsilon}^{r_H+\varepsilon+h}\frac{R^2}{e^{2U}}\eta(r)dr, \tag{13}$$

where $\omega = 2$ is the spin degeneracy of the particles of the fields considered here and

$$\eta(r) = e^{2U}\sum_p B(r)-\frac{2s}{R^2} = = 4\left[-(s^2+1)\frac{R''}{R}+\frac{R'^2}{R^2}\right]e^{2U}+4s^2(e^{2U})'\frac{R'}{R}-\frac{2s}{R^2}. \tag{14}$$

Then we obtain the total entropy and energy by

$$S = \beta^2\frac{\partial F}{\partial\beta} = \frac{15+(-1)^{2s}}{90}\frac{\omega\pi^3}{\beta^3}\int_{r_H+\varepsilon}^{r_H+\varepsilon+h}\frac{R^2}{e^{4U}}dr+ +\frac{3+(-1)^{2s}}{12}\frac{\pi}{\beta}\int_{r_H+\varepsilon}^{r_H+\varepsilon+h}\frac{R^2}{e^{2U}}\eta(r)dr, \tag{15}$$

$$U=\frac{\partial(\beta F)}{\partial\beta}=\frac{15+(-1)^{2s}}{120}\frac{\omega\pi^3}{\beta^4}\int\limits_{r_H+\varepsilon}^{r_H+\varepsilon+h}\frac{R^2}{e^{4U}}dr+$$
$$+\frac{3+(-1)^{2s}}{24}\frac{\pi}{\beta^2}\int\limits_{r_H+\varepsilon}^{r_H+\varepsilon+h}\frac{R^2}{e^{2U}}\eta(r)dr.$$

On the other hand, the total entropy and the total energy are given by

$$S=\int\limits_{r_H+\varepsilon}^{r_H+\varepsilon+h}\delta(r)\frac{A(r)dr}{e^U}=\int\limits_{r_H+\varepsilon}^{r_H+\varepsilon+h}\sigma(r)\frac{4\pi R^2dr}{e^U},$$
$$U=\int\limits_{r_H+\varepsilon}^{r_H+\varepsilon+h}\rho(r)A(r)dr=\int\limits_{r_H+\varepsilon}^{r_H+\varepsilon+h}\rho(r)4\pi R^2dr. \quad (16)$$

where we have taken the spherical shell as the volume element and $A(r)=\int\sqrt{g_{\theta\theta}g_{\varphi\varphi}}d\theta d\varphi$ is the area of the curved surface at random point r outside the horizon in the spacetime (1). The factor $1/\sqrt{g_{tt}}=e^{-U}$ does not appear in the integral for the total energy of the thermal excitations [23]. By comparing Eq. (16) with Eq. (15), we obtain the entropy density and energy density

$$\sigma(r)=\frac{15+(-1)^{2s}}{360}\omega\pi^2T^3(r)+\frac{3+(-1)^{2s}}{48}\eta(r)T(r), \quad (17)$$

$$\rho(r)=\frac{15+(-1)^{2s}}{480}\omega\pi^2T^4(r)+\frac{3+(-1)^{2s}}{96}\eta(r)T^2(r), \quad (18)$$

where $T(r)=\beta^{-1}e^{-U}$ is the local temperature [24].

Using the Unruh-Wald relation $P(r)=\sigma(r)T(r)-\rho(r)$ [25], we can obtain the pressure

$$P(r)=\frac{15+(-1)^{2s}}{1440}\omega\pi^2T^4(r)+\frac{3+(-1)^{2s}}{96}\eta(r)T^2(r). \quad (19)$$

The equation of state is

$$\rho(r)-3P(r)=-\frac{3+(-1)^{2s}}{48}\eta(r)T^2(r), \quad (20)$$

which is not zero for the spin fields.

4. Conclusion. We have used the WKB approximation to evaluate the entropy density, energy density, pressure and equation of state for the perfect relativistic gases of massless particles with spin $s=1/2$, 1, 3/2 and 2 around spherically symmetric and static geometries, which are given by Eqs. (17)–(20). As previous analysis, our results show that the quantities for spin fields contain an additional term with spin-dependence. This is very different from the usual result that the entropy density, energy density and pressure take the same forms as that in flat spacetime. The result is compatible with the conclusions that black hole entropy is not exactly proportional to the horizon area [8–12] and that Hawking radiation is not purely thermal radiation [3–6], and it is important for further discussion of black hole properties.

For the Reissner–Nordström black hole,

$$e^{2U}=1-\frac{2M}{r}+\frac{Q^2}{r^2};\quad R=r, \quad (21)$$

then

$$\eta(r)=\frac{4(1-2s^2)Q^2}{r^4}+\frac{8(s^2-1)M}{r^3}+\frac{4-2s}{r^2}. \quad (22)$$

For the Garfinkle-Horowitz-Strominger dilatonic black hole with a dilaton parameter a,

$$e^{2U}=1-\frac{2M}{r};\quad R^2=r(r-a), \quad (23)$$

hence

$$\eta(r)=\frac{(2r-a)^2}{4r^2(r-a)^2}+(1+s^2)\frac{a^2(r-2M)}{r^3(r-a)^2}+$$
$$+\frac{4s^2M(2r-a)}{r^3(r-a)}-\frac{2s}{r(r-a)}. \quad (24)$$

Equations (22) and (24) show that the subleading terms usually decrease as $1/r^2$ or more rapidly for large r, and therefore the terms can be neglected more frequently. However, we also note that the term is important and cannot be neglected at sufficiently low temperatures, for example near-extremal black hole.

The author would like to thank the reviewers very much for their helpful comments.

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