$$\frac{c}{c_0} = \frac{t}{t_0} = \exp\{-\frac{1}{8}(\xi_0^2 - \xi^2)\} \tag{6}$$

in place of the previous

$$\frac{c}{c_0} = \frac{t}{t_0} = \exp\{-A^2(\xi_0 - \xi)\}. \tag{6a}$$

Thus, the difference in the character of the long epochs in both cases reduces only to a different connection between the time t and the variable ξ, governing the oscillations of the functions (5). If t_0 and t_1 are the upper and lower time limits of the long epoch, then $8\ln(t_0/t_1) \approx \xi_0^2$ in the case of (6) and $A^{-2}\ln(t_0/t_1) \approx \xi_0$ in the case of (6a). On the other hand, the value of ξ_0 determines the total number of oscillations during the long epoch (equal to $\xi_0/2\pi$). It is clear therefore that at a specified ratio t_0/t_1 the number of oscillations is in general smaller in the case of (6) than in the case of (6a).

In connection with the foregoing, we can make the following two remarks.

1. The distinguishing feature of the type-IX model, compared with type VIII, is that when $\lambda = \mu = 1$ the difference $\lambda a^2 - \mu b^2$ in Eqs. (1) is small together with the difference $a - b$; such a contraction requires not only that the signs of λ and μ be identical, but also that these quantities be essentially constant. One can therefore expect, in the most general case of an inhomogeneous space metric, the character of its time dependence during long epochs will correspond to (6) and not to (6a). This conclusion is indeed confirmed by an analytic construction of a general solution for a long epoch, as will be shown elsewhere by V. A. Belinskii and I. M. Khalatnikov.

2. A homogeneous space of type VIII has an infinite volume, whereas a type-IX space is closed. Therefore the aggregate of these two examples is evidence of the absence of a direct connection between the oscillatory approach to the singular point and the openness or closedness of the model.

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STRUCTURE OF TURBULENT-VISCOSITY COEFFICIENT FOR AN ISOTROPIC TURBULENCE

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An isotropic-turbulence equation, containing turbulent viscosity, was proposed in [1]. By introducing the turbulent viscosity, various hypotheses regarding the closing of the system of equations of isotropic turbulence are replaced by some hypotheses concerning the structure of the coefficient of turbulent viscosity, defined by

$$\nu_T = k(r^2/t). \tag{1}$$

The quantity k may depend in general on the Reynolds number of the turbulence. This hypothesis has a number of theoretical advantages over the formula proposed in [1]:

$$v_T = k[B_d^d(0, t)]^{1/2} r, \tag{2}$$

and ensures, in particular, the required behavior of B_d^{dd} as $r \to 0$. The value of the coefficient k in (1) can vary in a wider range than in (2). In this case the principal equation of isotropic turbulence is written in the form

$$\frac{\partial B_d^d}{\partial t} = \frac{1}{r^4} \frac{\partial}{\partial r} r^4 (2\nu + 2k \frac{r^2}{t}) \frac{\partial B_d^d}{\partial r}, \tag{3}$$

which is equivalent to the hypothesis that the third moments are written in the form

$$B_d^{dd} = 2k \frac{r^2}{t} \frac{\partial}{\partial r} B_d^d. \tag{4}$$

We note first that formula (4), based on the use of hypothesis (1), ensures satisfaction of all the requirements that the function B_d^{dd} must satisfy in accord with its definition. Thus, this hypothesis not only yields the structure of the viscosity coefficient, but also makes it possible to regard Eq. (3) as the first equation of an infinite system for the correlation functions. Equation (3) has a self-similar solution in the form

$$B_d^d(r, t) = t^n u(\eta), \text{where } \eta = r t^{\beta}, \quad \beta = -\frac{1}{2}.$$

When $n = 5\beta$, the equation for $u(\eta)$ can be integrated in final form:

$$u(\eta) = \frac{c}{(1 + k\eta^2/\nu)^{1/8k}}.$$

The integral invariant of the isotropic turbulence

$$\int_0^\infty B_d^d(r, t) r^4 dr = \text{const}$$

is finite for $n = 5\beta$ when $k < 1/20$.

In this case we have $n = -5/2$ for $\beta = -1/2$, and consequently

$$B_d^d = \frac{c}{t^{5/2}} \frac{1}{(1 + k\eta/\nu)^{1/8k}}.$$

By taking the limit as $k \to 0$ in this formula we obtain the well-known equation obtained in [2], which holds when only molecular viscosity is in action:

$$B_d^d = \frac{c}{t^{5/2}} \exp(-\frac{r^2}{8\nu t}).$$

If the molecular viscosity is negligibly small compared with the turbulent viscosity, then Eq. (3), which takes the form

$$\frac{\partial B_d^d}{\partial t} = \frac{1}{r^4}\frac{\partial}{\partial r}\left(2k\frac{r^6}{t}\frac{\partial}{\partial r}B_d^d\right)$$

has a self-similar solution

$$B_d^d(r, t) = t^n v(\eta), \quad \eta = rt^{\beta}$$

with arbitrary exponent β.

It follows therefore that in the presence of both types of viscosity, molecular and turbulent, with the turbulent viscosity predominating, it is advisable to seek an approximate solution of (3) with an arbitrary exponent β. The exact equation (3) is then replaced by an approximate one for the "quasi-self-similar" regime. In the latter equation, the term containing the molecular viscosity has a time factor $t^{2\beta+1}$, which can be replaced by its mean value $s = \overline{t^{2\beta+1}}$ in the given interval of t.

The approximate equation for the "quasi-self-similar" regime takes the form (for $n = 5\beta$)

$$\beta\frac{d}{d\eta}(\eta^5 v) = \frac{1}{\eta^4}\frac{d}{d\eta}\,\eta^4[2\nu s + 2k\eta^2]\frac{dv}{d\eta}.$$

Its solution is

$$v = c(a^2 + \eta^2)^{\beta/4k} \text{ and } B_d^d(r, t) = ct^{5\beta}(a^2 + \eta^2)^{\beta/4k},$$

where $\alpha^2 = \nu s/k$. This solution is valid of $\beta/2k < -5$.

In accordance with Eq. (4), the formula for the third moments then takes the form

$$B_d^{dd}(r, t) = \beta\frac{\eta^3}{a^2 + \eta^2}B_d^d(r,t)\,t^{-(1+\beta)}.$$

For the normalized function B_d^{dd} we get

$$\frac{B_d^{dd}(r,t)}{[B_d^d(0, t)]^{3/2}} = \frac{\beta}{\sqrt{c}}\frac{\eta^3}{a^2 + \eta^2}\frac{B_d^d(r, t)}{B_d^d(0, t)}\,t^{-(1+\frac{7}{2}\beta)}. \tag{5}$$

The case $\beta = -1/2$ ($s = 1$) corresponds to an exact solution of (3). This solution with an arbitrary exponent β permits a better description of the experimental data, which deviate appreciably from self-similarity with respect to the parameter $\eta_1 = r/\sqrt{\nu t}$. If $\beta \neq -2/7$, then the time variation of the minimum of (5) is obtained without the additional assumption that k depends on the time.

Preliminary data on computer solution of Eq. (3) have shown that sufficiently accurate results can be obtained with the aid of the "quasi-self-similar" solution.

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