GROUP INTERPRETATION OF THE COMPLEX SPIN

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Submitted 28 July 1968

The Regge-pole theory has brought into being the concept of particles with complex spin, which are regarded either as virtual or as unstable. The notion of a complex spin may also be of interest from an entirely different point of view: it is possible that "particles" with complex spin, which have properties that exclude their direct observability, may serve as "material" for the construction of real particles (i.e., play the role of quarks)[1]. In order to make a quantum-mechanical consideration of such "particles" possible, a group interpretation of the complex spin is necessary.

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[1] This idea was advanced by I. S. Shapiro

Several recent papers are devoted to searches for such an interpretation. These papers considered either a local representation (R) of the SU(2) group (cf., e.g., [1 - 3]) or the R of the Lie algebra of this group (e.g., [4]), or else an irreducible representation (IR) of the SU(1, 1) group (e.g., (5, 6]). However, none of these construction yields the law governing the transformation of the state of a "particle" with a complex spin under arbitrary rotation. A possible way out might be the consideration of a family of local R compatible with one another and covering as an aggregate the entire SU(2) group. In this paper, however, we propose another scheme.

In our scheme, as in those propsed earlier, it is impossible to specify the law of transformation under arbitrary rotation satisfied by the spin[2] wave function of a "particle" with unphysical spin (i.e., one not equal to an integer or a half-integer); therefore the wave function should not be considered at all in this scheme. However it is possible to introduce bilinear quantities, and in many cases it is sufficient to consider just such quantities. We can specify also their transformational properties under arbitrary rotation, generalizing the law of transformation of the bilinear quantities at physical values of the spin. Such bilinear quantities include, for example, the density matrix and the Green's function of one particle and the wave function of a system of two particles whose spins differ by an integer or a half-integer; it is possible to construct from the same quantities the amplitudes of processes in which "particles" with complex spin take part.

The possibility of such a construction is based on the fact that, on the one hand, the bilinear quantities transform in accord with a representation that is the direct product of two IR, and on the other hand it is possible to extend the concept of a direct product of two IR of weights s_1 and s_2 (but not of an individual IR) to include complex s_1 and s_2 (if s_1 and s_2 is an integer or half-integer). We shall show how to carry out such an extention for the case $s_1 = s_2$, namely, we shall consider the polarization matrix of a "particle" with complex spin s.

If we realize IR of the SU(2) group of weight s in the space of polynomials of the complex variable z of degree not higher than 2s, then the operator of the respresentation will act in accordance with the formula

$$\hat{T}_u^{(s)} \Psi(z) = (\beta z + \tilde{a})^{2s} \Psi\left(\frac{a z - \bar{\beta}}{\beta z + \bar{a}}\right),$$ (1)

$$u = \begin{pmatrix} a & \beta \\ -\bar{\beta} & \bar{a} \end{pmatrix} \in SU(2).$$

An attempt of a direct extension of the transformation law (1) to unphysical s encounters the following obstacle: if 2s is non-integer the expression $(\beta z + \bar{a})^{2s}$ has no unique meaning. In the same realization, the density matrix

$$\rho(z, \bar{z}') = \sum_{n=1}^{2s+1} w_n \, \Psi_n(z) \overline{\Psi_n(z')}$$ (2)

(where $\Psi_n(z)$ - basis functions, say the eigenfunctions of the spin-projection operator)

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is a function of z and $\bar{z}'$, which, by virtue of its polynomial character, can be uniquely reconstructed from its values $\rho(z, z)$ at $z' = z$. It follows from (1) that the law of transformation of the function $\rho(z, \bar{z})$ is

$$(\hat{T}_u^{(s)} \otimes \hat{T}_u^{(s)*}) \rho(z, \bar{z}) = [(\beta z + \bar{a})(\bar{\beta} z + a)]^{2s} \rho\left(\frac{az - \bar{\beta}}{\beta z + \bar{a}}, \frac{\bar{a}\bar{z} - \beta}{\bar{\beta}\bar{z} + a}\right) \tag{3}$$

Here we have raised to the power 2s a real non-negative number; the result of such an operation is uniquely defined for any complex s, and therefore formula (3) can be accepted as the transformation law for the polarization density matrix of a "particle" with arbitrary complex spin.

From the group-theoretical point of view formula (3) can serve as a definition of the operator of the representation $T^{(s)} \otimes T^{(s)*}$, which is a product of an IR of complex weight s by its conjugate[3]; this "product," cannot be factorized. To find the aggregate of the functions making up to space of the representation $T^{(s)} \otimes T^{(s)*}$ we note that at integer values of 2s this representation is a narrowing of the group SL(2, C) of weight (2s, 2s) on the SU(2) group. Extending this statement to complex s (it now serves as a definition of what is meant by the representation $T^{(s)} \otimes T^{(s)*}$, we arrive again at the transformation law (3), and also to a definition of the space D of the representation $T^{(s)} \otimes T^{(s)*}$, namely, it coincides with the space of the IR of group SL(2, C), i.e., it consists of the functions $\rho(z, \bar{z})$, which are infinitely differentiable in the entire plane together with their inversions $\overset{\vee}{\rho}(z, z) \equiv (z\bar{z})^{2s} \rho(-1/z, -1/\bar{z})$.

We can again interpret as density matrices those of the functions $\rho(z, \bar{z})$ which satisfy a number of conditions (extending the conditions of normalization, hermiticity, and non-negativity); an example is the density matrix of a "particle" with magnetic moment μ, which is in thermodynamic equilibrium in a magnetic field:

$$\rho(z, \bar{z}) = \frac{sh(\mu H/2T)}{sh[(2s + 1)\mu H/2T]} (e^{\mu H/2T} + z\bar{z}e^{-\mu H/2T})^{2s}. \tag{4}$$

We can also specify a method of calculating the mean values of the "observables" in each of such states; for example,

$$\langle s_z \rangle = \frac{2s + 1}{\pi} \int \frac{d^2 z}{(1 + z\bar{z})^{2s + 2}} \left(s\rho(z, \bar{z}) - z \frac{\partial \rho(z, \bar{z})}{\partial z}\right). \tag{5}$$

We calculate analogously the mean values of other spin projections or of any polynomial of the spin-projection operators.

However, a direct calculation of the probabilities of different "measurement results" is impossible, for this would require projection of the given density matrix on a pure state, and there is no place for pure states in the space D. For example, a pure state with projection of the spin σ on the z axis should be described by a density matrix

$$\rho(z, \bar{z}) = \frac{\Gamma(2s + 1)}{\Gamma(s - \sigma + 1)\Gamma(s + \sigma + 1)} (z\bar{z})^{s - \sigma}, \tag{6}$$

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[3] We note that the IR $T^{(s)*}$ is equivalent to the representation $T^{(s)}$.

but this function does not belong to D for any value of σ. Therefore the terms "density matrix" and "mean value" are to a considerable degree arbitrary.

We note that the representations $T^{(s)} \otimes T^{(s)*}$ introduced above are equivalent to one another at all unphysical s, namely, each of them is an infinite direct sum of all the single-valued IR of a rotation group

$$T^{(s)} \otimes T^{(s)^*} = \sum_{l=0}^{\infty} \oplus T^{(l)} , \tag{7}$$

but the formulas for the calculations of the mean values are different for different s (cf., e.g., [5]).

An analogous construction can be presented also for other quantities that are bilinear in the wave function. For example, functions from the space D can be interpreted not as density matrices of one particle, but as spin wave functions of a system of two particles with spin s. It then follows from the expansion (7) that such a system can be in a state with arbitrary integer value of the total spin.

A more detailed exposition of this scheme will be the subject of another paper.

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