QUANTUM-MECHANICAL FORMATION OF VORTICES IN A SUPERFLUID LIQUID

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We consider here the formation of vortices in a superfluid liquid that moves relative to the vessel walls, at T = 0. Since there is no normal component, the theory of fluctuation formation of vortices [1] is not applicable. The principal role is played in this case by the interaction between the liquid and the vessel walls. As a result of this interaction, the state of homogeneous motion of the liquid can go over quantum-mechanically into a state of motion with the vortex, at the same total kinetic energy of the liquid. The interaction of the liquid with the wall is caused by inhomogeneities on the vessel wall, which perburb the homogeneous motion of the liquid. The possibility of such a vortex-formation mechanism was first pointed out by Vinen [2]. A similar quantum-mechanical transition from one macroscopic state into another was considered in [3, 4], where nucleus formation in the phase transition by tunneling was described. In this case, however, suprabarrier reflection takes place instead of subbarrier tunneling.

We consider inhomogeneities having a characteristic dimension R >> a, where a is the radius of the vortex core and is of the order of atomic dimension. This makes it possible to use a hydrodynamic description. We assume for convenience an inhomogeneity in the form of a hemisphere of radius R on a flat vessel wall, over which liquid flows with velocity u at infinity. We seek the probability of formation of a vortex in the form of a half-ring whose ends glide freely over the wall. When the vortex moves, the plane of the half-ring remains continuously perpendicular to u, and the ring axis passes through the center of the sphere.

Let r be the radius of the ring and z the coordinate of its plane, reckoned from the center of the sphere in the direction of u. The equations of motion of the vortex are then (cf., e.g., [5])

$$\frac{dr}{dt} = -\frac{1}{\pi\rho\kappa r}\frac{\partial E}{\partial z}, \qquad \frac{dz}{dt} = \frac{1}{\pi\rho\kappa r}\frac{\partial E}{\partial r}, \tag{1}$$

where κ is the circulation of the velocity, ρ the liquid density, and E the total kinetic energy of the liquid with the vortex. In this geometry it is easy to calculate E, since mirror reflection yields a ring and a sphere having a common axis [6]:

$$E(r, z) = E_0 + \frac{1}{4}\rho\kappa^2\left\{r\left(\ln\frac{8r}{a} - 2\right) - rQ_{1/2}\left[1 + \frac{1}{2}\left(\frac{r^2 + z^2 - R^2}{rR}\right)^2\right] + \frac{2\pi u}{\kappa}r^2\left[1 - \left(\frac{R^2}{r^2 + z^2}\right)^{3/2}\right]\right\}, \tag{2}$$

where $Q_{1/2}$ is a spherical function of the second kind and E_0 the kinetic energy of the liquid in the absence of the vortex.

The E(r, z) = const curves are the vortex motion trajectories. Vortex trajectories with κ < 0 are shown in Fig. 1. The arrows show the vortex motion directions. The thick lines are the trajectories $r_1(z)$ and $r_2(z)$ with $E = E_0$.

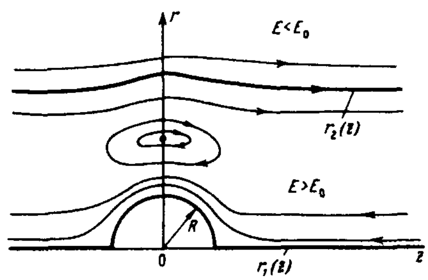

Fig. 1

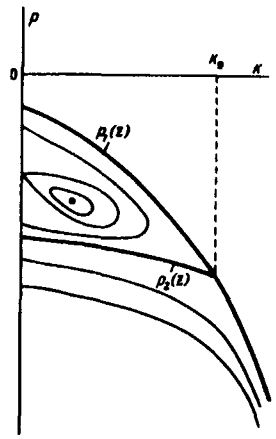

Fig. 2

The trajectory $r_1(z)$ passes over the surface of the sphere and describes outside this surface the motion of a ring with zero radius ($r \sim a$). This trajectory corresponds to the liquid motion in the absence of vortices. It is required to find the probability of the transition from the trajectory $r_1(z)$ to the trajectory $r_2(z)$.

To this end, we use the circumstance that Eqs. (1) are of Hamiltonian form. Indeed, if we introduce $p = (\pi/2)\rho\kappa r^2$, then we get from (1)

$$\frac{dz}{dt} = \frac{\partial E}{\partial p}, \qquad \frac{dp}{dt} = -\frac{\partial E}{\partial z}. \tag{3}$$

Consequently, p and z are cononically conjugate variables. We can therefore introduce the Hamiltonian operator $H(p, z) = E(r(p), z)$ and go over to the one-dimensional quasiclassical problem of the transition from trajectory $p_1(z)$ to trajectory $p_2(z)$ by suprabarrier reflection. To solve this problem we must find the classical transition point in the complex z plane. Obviously, since the Hamiltonian is even in z, this point is located on the imaginary axis. The thick lines show the trajectories $p_1(ik)$ and $p_2(ik)$ with energies $E = E_0$. Their intersection k_0 is the transition point. The transition probability is [7]

$$w = A\exp\left\{-\frac{2\operatorname{Im} S}{\hbar}\right\} = A\exp\left\{-\frac{2}{\hbar}\int_0^{k_0} dk\left[p_1(ik) - p_2(ik)\right]\right\}, \tag{4}$$

where S is the action along the transition line.

We can calculate ImS in two limiting cases:

$$\operatorname{Im} S = \frac{\rho\kappa^2 R^2}{6u}\left(\ln\frac{|\kappa|}{ua}\right)^2, \qquad \frac{|\kappa|}{R} \ll u \ll \frac{|\kappa|}{a}; \tag{5}$$

$$\operatorname{Im} S = \frac{\rho\kappa^4}{24\pi^2 u^3}\left(\ln\frac{|\kappa|}{ua}\right)^3, \qquad u \ll \frac{|\kappa|}{R}.$$

In the general case it is impossible to calculate the pre-exponential factor A in (4). To estimate this coefficient, we note that the motion of the liquid is not potential in a layer of approximate width a next to the wall. This solenoidal layer can be replaced by an assembly of vortices with radii on the order of a, distributed with a surface density $n \sim u/a|\kappa|$. The factor A is proportional to the flux of these vortices, to the cross section for their scattering by the inhomogeneity, and to the number N of the inhomogeneities, i.e., $A \sim u^2 RN/a|\kappa|$.

ImS is independent of the radius R when $u \ll |\kappa|/R$. It is necessary to ascertain whether this quantity depends on the shape of the inhomogeneity. If the shape of the inhomogeneity is given by $r^2 + \beta^2 z^2 = R^2$, then the calculation becomes much more complicated. When $u \ll |\kappa|/R$, however, it is possible to determine the principal term of ImS from the following considerations: The trajectory $r_1(z)$ passes over the surface of the imhomogeneity and, consequently its equation is $r^2 + \beta^2 z^2 = R^2$. From this, going over to the imaginary axis $z = ik$, we obtain $p_1(z) = (\pi/2)\rho\kappa(R^2 + \beta^2 k^2)$. $p_2(z)$ depends little on z at small u, and can therefore be regarded as constant on the imaginary axis:

$$p_2(z) \approx \frac{\pi}{2} \rho \kappa \left(\frac{\kappa}{2\pi u} \ln \frac{|\kappa|}{ua} \right)^2.$$

The trajectories intersect at the point

$$k_o \approx \frac{|\kappa|}{2\pi u \beta} \ln \frac{|\kappa|}{ua} .$$

From this we get (see (4)):

$$\mathrm{Im}\, S \approx \frac{\rho \kappa^4}{24\pi^2 u^3 \beta} \left(\ln \frac{|\kappa|}{ua} \right)^3. \quad (6)$$

Thus, the inhomogeneity shape most favorable for vortex formation is that of half a flat disk ($\beta \to \infty$).

In conclusion, the author thanks S.V. Iordanskii for suggesting the problem and directing the work.

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DESTRUCTIVE INTERFERENCE IN THE SCATTERING OF FAST ELECTRONS IN A SINGLE CRYSTAL

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1. As is well known, elastic scattering in a single crystal exhibits interference properties even for relativistic electrons of small wavelength, inasmuch as the wavelength transmitted in the longitudinal direction (along the particle motion) ($\hbar = c = 1$, $\kappa = me^2 Z^{1/3}$)

$$(p_{1x} - p_{2x})^{-1} \sim (p\theta^2)^{-1} \sim p\kappa^{-2}$$

can greatly exceed the interatomic distances. Ter-Mikaelyan [1] investigated this effect in the Born approximation for scattering by a single crystal as a whole. The condition for the applicability of perturbation theory

$$\frac{1}{v} \int dx\, U \approx \frac{1}{v} UL \ll 1 \quad (1)$$

($v = p/E$ is the velocity of the relativistic electron) imposes a rigid upper limit on the thickness L of the single crystal in the electron motion direction. The purpose of the present communication is to call attention to the fact that