ASYMPTOTIC LOWER BOUND FOR THE SCATTERING-AMPLITUDE PHASE IN THE t-PLANE

A. N. Tverdokhlebov
All-union Institute of Scientific and Technical Information, USSR Academy of Sciences, Physics Division
Submitted 24 March 1969
ZhETF Pis. Red. 9, No. 9, 537-542 (5 May 1969)

We have obtained the following property, asymptotic for real $s \to \infty$, of the phase of the scattering amplitude $A(t, s)$: for any single-valued, possibly discontinuous, definition of the argument $\phi(t, s) = \arg\{A(t, s)\}$ in the t-plane, and for real s, the maximum $\alpha(s) = \max\{\phi(t, s)\}$ is bounded from below when $|t| < s$ by

$$\alpha^2(s) \gtrsim \frac{\pi\mu^2}{8} \cdot \frac{\sigma_{tot}^2(s)}{\sigma_{el}(s)}, \tag{1}$$

where $\sigma_{tot}(s)$ and $\sigma_{el}(s)$ are respectively the total elastic scattering cross sections of particles of mass μ. Since present notions concerning the behavior of the total and elastic cross sections, indicate that $\mu^2\sigma^2_{tot}(s)/\sigma_{el}(s) \to \infty$, this property means infinite oscillations of the amplitude asymptotically as $s \to \infty$. The following analytic properties are proposed for the amplitude $A(t, s)$: 1) $A(t, s)$ is analytic in t for all s outside the real cuts $(4\mu^2, +\infty)$ and $(-\infty, -s)$ in the t-plane; 2) the modulus $|A(t, s)|$ increases more slowly than any exponential in t; 3) $\alpha(s) = o(s)$ as $s \to \infty$.

The inequality (1) will be derived in the following fashion. We represent the amplitude $A(t, s)$ in the form

$$A(t,s) = P_N(t, s)\frac{A(t, s)}{P_N(t, s)} = P_N(t, s)\exp\left\{\ln\frac{A(t, s)}{P_N(t, s)}\right\},$$

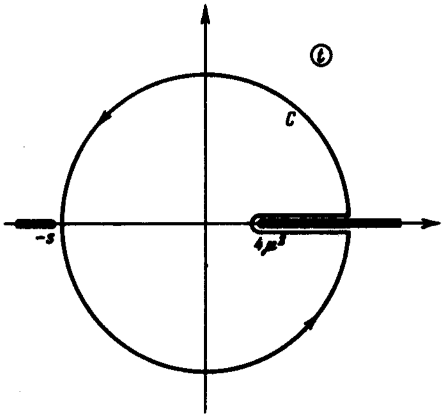
$P_N(t, s)$ is a polynomial in t with roots $t_k(s)$ coinciding with the zeros of $A(t, s)$ in the circle $|t| < s$, i.e.,

$$P_N(t, s) = \prod_{k=1}^{N}(1 - t/t_k(s)).$$

The function $\ln\{A(t, s)/P_N(t, s)\}$ is analytic in the circle $|t| < s$, with a cut, and a dispersion relation can be written for it. The integral over the circle then tends to zero as $s \to \infty$, and then modulus of the amplitude $|A(t, s)|$ is expressed, at physical values of t in the s-channel, in terms of the modulus of the polynomial $P_N(t, s)$ and the integral of the function $\phi(t, s) = \arg\{A(t, s)\}$ along the cuts. We then obtain from the fact that the phase is bounded $(\phi(t, s) < \alpha(s))$, the lower bound for $\sigma_{el}(s)$ for an arbitrary polynomial $P_N(t, s)$.

Let $t_k(s)$ be the zeros of $A(t, s)$, $N = N(s)$ the number of zeros in the circle $|t| < s$, and $P_N(t, s) = \prod_{k=1}^{N}(1 - t/t_k(s))$ a polynomial in t. We write down the Cauchy integral for the function $1/t \ln\{A(t, s)/P_N(t, s)\}$, which is analytic in the circle $|t| < s$ with a cut $(4\mu^2, s)$ (the integration contour C is shown in the figure)

$$\frac{1}{t}\ln\left\{\frac{A(t, s)}{P_N(t, s)}\right\} = \frac{1}{t}\ln\left\{\frac{A(0, s)}{P_N(0, s)}\right\} + \frac{1}{2\pi i}\oint_C \frac{\ln\{A(\zeta, s)/P_N(\zeta, s)\}}{\zeta(\zeta - t)}d\zeta =$$



By virtue of assumptions 2) and 3), the integral over the circle $|\zeta| = s$ is $o(s)$ as $s \to \infty$, it is necessary to take into account here the fact that $\arg\{P_N(\zeta, s)\} \lesssim 2\alpha(s) = o(s)$, since the number of zeros of ten amplitude, and consequently of the polynomial, does not exceed $\alpha(s)/\pi$ in principle. Thus, recognizing that $P_N(\zeta, s)$ is analytic on the cut, we obtain

$$\ln\left\{\frac{A(t, s)}{A(0, s)P_N(t, s)}\right\} \simeq \frac{t}{2\pi i}\int_{4\mu^2}^{s}\frac{\ln|A(\xi + i\epsilon, s)/A(\xi - i\epsilon, s)|\,d\xi}{\xi(\xi - t)} + \frac{t}{2\pi}\int_{4\mu^2}^{s}\frac{\arg\{A(\xi + i\epsilon, s)\} - \arg\{A(\xi - i\epsilon, s)\}}{\xi(\xi - t)}d\xi$$

(we used $P_N(0, s) = 1$).

For physical values of t ($\mathrm{Im}\, t = 0$, $-s < \mathrm{Re}\, t < 0$) we obtain a representation of the modulus of the amplitude $|A(t, s)|$ in terms of the integral of its argument

$$|A(t, s)| = |A(0, s)|\,|P_N(t, s)|\exp\left\{\mathrm{Re}\left[\ln\frac{A(t, s)}{A(0, s)P_N(t, s)}\right]\right\} =$$

$$= |A(0, s)|\,|P_N(t, s)|\exp\left\{\frac{t}{2\pi}\int_{4\mu^2}^{s}\frac{\arg\{A(\xi + i\epsilon, s)\} - \arg\{A(\xi - i\epsilon, s)\}}{\xi(\xi - t)}d\xi + o(s)\right\}. \tag{2}$$

Let us write out an expression for the elastic and total cross sections in terms of the amplitude

$$\sigma_{el}(s) = \frac{1}{16\pi s(s - 4\mu^2)}\int_{-(s-4\mu^2)}^{0}|A(t, s)|^2 dt, \tag{3}$$

$$\sigma_{tot}(s) = \frac{\mathrm{Im}\{A(0, s)\}}{\sqrt{s(s - 4\mu^2)}} \simeq \frac{\mathrm{Im}\{A(0, s)\}}{s}, \tag{4}$$

$$\sigma_{el}(s) \gtrsim \frac{1}{16\pi}\sigma_{tot}^2(s)\int_{-(s-4\mu^2)}^{0}\left|\frac{A(t, s)}{A(0, s)}\right|^2 dt. \tag{5}$$

From (2), recognizing that $t < 0$, we obtain

$$\left|\frac{A(t, s)}{A(0, s)}\right|^2 \geqslant |P_N(t, s)|^2\exp\left\{\frac{2\alpha(s)t}{\pi}\int_{4\mu^2}^{s}\frac{d\xi}{\xi(\xi - t)}\right\} \simeq$$

$$\simeq |P_N(t, s)|^2\exp\left\{\frac{2\alpha(s)}{\pi}\ln\frac{4\mu^2}{4\mu^2 - t}\right\} \geqslant |P_N(t, s)|^2\exp\left\{\frac{2\alpha(s)}{\pi}\frac{t}{4\mu^2}\right\}. \tag{6}$$

We put

$$t = -4\mu^2 x,\ t_K(s) = -4\mu^2 x_K(s),\ K = 2a(s)/\pi,$$

$$|P_N(x)|^2 \equiv |P_N(-4\mu^2 x, s)|^2.$$

Then, substituting (6) in (5), we get

$$\sigma_{el}(s) \gtrsim \frac{\mu^2}{4\pi}\sigma_{tot}^2(s)\int_0^{(s/4\mu^2)^{-1}} |P_N(x)|^2 e^{-Kx}dx \propto \frac{\mu^2}{4\pi}\sigma_{tot}^2(s)\times$$

$$\times\left[\int_0^\infty |P_N(x)|^2 e^{-Kx}dx + 0(e^{-Ks/4\mu^2})\right].$$

The integral

$$I(s) = \int_0^\infty |P_N(x)|^2 e^{-Kx}dx$$

is bounded from below, since $|P_N(x)|^2$ is a positive-definite polynomial of degree 2N, so that $P_N(0) = 1$. Let

$$|P_N(x)|^2 = \sum_{m,n=0}^{N} a_n a_m x^n x^m,\quad a_0 = 1.$$

To obtain min {I(s)} for arbitrary a_n we can assume that all the zeros of x_K are real and consequently the coefficients a_n are also real: $a_n^* = a_n$.

If we calculate the integral I(s) in explicit form in terms of the coefficients a_n, then the condition on min I(s) constitute a system of N linear equations in a_n:

$$\frac{\partial I(s)}{\partial a_n} = 0,\ n = 1,\ldots, N.$$

Taking these equations into account in the expression for I(s), we express

$$b \equiv \min I(s)$$

linearly in terms of a_n. We thus obtain a system of N + 1 linear equation relative to the N + 1 unknowns a_n and b. Solving this system with respect to b, we get

$$b = \frac{1}{K(N+1)}$$

and consequently

$$I(s) \geqslant \frac{1}{K(N+1)} \simeq \frac{\pi}{2a(s)}\frac{\pi}{(a(s)+\pi)},$$

since N, in accordance with the principle of the argument, does not exceed $\alpha(s)/\pi$ (see above). Consequently

$$\sigma_{el}(s) \gtrsim \frac{\mu^2\pi}{8}\frac{\sigma_{tot}^2(s)}{a(s)[a(s)+\pi]},$$

or

$$\alpha(s)[\alpha(s)+\pi] \gtrsim \frac{\pi}{8}\mu^2 \frac{\sigma_{tot}^2(s)}{\sigma_{el}(s)}.$$

If we assume that

$$\frac{\mu^2 \sigma_{tot}^2(s)}{\sigma_{el}(s)} \to \infty \quad \text{as } s \to \infty$$

then $\alpha(s) \to \infty$ and we obtain inequality (1) for the lower asymptotic bound of the maximum $\alpha(s)$ of the argument of the amplitude $A(t, s)$ in the t-plane.