CHANGE OF VOLUME OF A FERMI SYSTEM WITH CHANGE IN THE NUMBER OF PARTICLES

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Submitted 14 July 1972
ZhETF Pis. Red. 16, No. 7, 410 - 414 (5 October 1972)

It is well known that the radius of a liquid drop increases with increasing number of particles like $R = r_0 A^{1/3}$. When even a relatively small number of particles, $K < A^{2/3}$ is added the volume density ρ_0 remains practically unchanged, and the entire change of $\rho(r)$ occurs on the surface, where $\delta\rho \sim (\partial\rho/\partial R)\delta R \sim k/A^{2/3}$ (the derivative $\partial\rho/\partial R$ does not contain the parameter $A^{-1/3}$). Numerous experiments have confirmed the $A^{1/3}$ law also for the dependence of the nuclear radius on the number of particles. However, if one simply states the sum over the states λ_i of the added quasiparticles (as is usually done in the shell model), then the result is entirely different, namely, the change of the density $\delta_0\rho(r) = \Sigma|\phi\lambda_i(r)|^2$ on the edge of the nucleus turns out to be of the order of K/A, i.e., smaller by a factor $A^{1/3}$ than in a liquid (as it should be for a gas of independent particles). Allowance for the polarization effect does not change this result in the general case, and a Fermi system of finite dimensions behaves like a gas of interacting quasiparticles, and not like a liquid. Arguments were therefore advanced that the change of the nuclear radius with changing number of particles is a jumplike process and that the jump occurs when the field of the additional particles becomes strong, i.e., comparable with the distance $\delta\varepsilon_{sp}$ between the combining single-particle levels ($\delta\varepsilon_{sp} \sim \varepsilon_F A^{-1/3}$), for which it is necessary to have $k \gtrsim A^{2/3}$.

There is, however, another possiblity. Indeed, the radius of the drop, at a given number of particles A is determined by the condition $(\partial E/\partial R)_A = 0$. Such a condition will be automatically satisfied if in the real system there exist low-lying surface 0^+ oscillations, for the description of which one can introduce the collective Hamiltonian

$$E(R) = E_0 + \frac{C_0}{2}(R - R_0)^2 + \frac{B_0\dot{R}^2}{2}. \tag{1}$$

For the Hamiltonian (1) to be meaningful, it is necessary to satisfy the adiabaticity conditions $\omega_0 \lesssim \delta\varepsilon_{sp}$. Let us ascertain when this is possible. The local amplitude F(r) of the quasiparticle interaction, which determines the rigidity C_0, varies rapidly (as do the density $\rho(r)$ and the mass operator $\Sigma(r)$) in the surface region of the system, and the particles are attracted on the edge. (For a nucleus, for example, $f^+_{ex} \simeq -4$, whereas $f^+_{in} \simeq \pm 0.2$ [1] ($f^+ = F^+ p_F M/\pi^2$; $F^+ = F^{pp} + F^{-np}$)). Therefore there can exist in the surface layer collective oscillations whose amplitude attenuates rapidly inside, where all the gradients are equal to zero (examples of oscillations of this type in ordinary hydrodynamics are internal waves in a heavy liquid [2]). The attraction on the edge leads to a decrease of the surface rigidity C_0, which can become anomalously small under certain conditions. Then even a weak action on the system will cause a considerable change in the surface density, a characteristic property of a liquid. We use in this paper the formalism and notation of the theory of finite Fermi systems [3]. The equation for the change $\delta\rho(r)$ following addition of certain quasiparticles in the state λ_i is

$$\delta\rho(\mathbf{r}) = \delta_0\rho(\mathbf{r}) + \int A(\mathbf{r}, \mathbf{r}'; \omega = 0) F(\mathbf{r}')\delta\rho(\mathbf{r}')d\mathbf{r}'. \tag{2}$$

The frequencies of the collective oscillations of the pole of the equation for the effective field are determined by the equation

$$g(\mathbf{r}) = F(\mathbf{r})\int A(\mathbf{r}, \mathbf{r}'; \omega_0) g(\mathbf{r}')d\mathbf{r}'. \tag{3}$$

In the symbolic notation which we shall frequently employ Eq. (2) takes the form $\delta\rho = \delta_0\rho + (AF\delta\rho)$ and (3) becomes $g = (FAg)$.

The propagator in (2) and (3)

$$A(r, r'\omega) = \frac{1}{2\pi i} \int G(r, r', \epsilon + \frac{\omega}{2}) G(r', r; \epsilon - \frac{\omega}{2}) d\epsilon \qquad (4)$$

can be represented as a sum of two components, a local one A^{ℓ}, which is a sharp peak of width $|\vec{r} - \vec{r}'| \sim r_0$, and a smooth long-range component $A \sim R^{-3}$ [3]. As seen from (3), A makes an appreciable contribution when g(r) differs from zero in the entire volume of the system. For the surface oscillations, g(r) has a sharp maximum in a narrow layer $\sim r_0$ and therefore the influence of the long-range action in A on these oscillations is negligible. As a rough approximation, the frequency ω_0 of the oscillations is a function of one dimensionless parameter $f_{ex} = F_{ex} p_F M/\pi^2$. With increasing $|f_{ex}|$, the frequency ω_0 decreases. In the adiabatic limit, when $\omega_0 < \delta\epsilon_{sp}$, the solutions of (2) and (3) can be interrelated by introducing a spherically-symmetrical "proper field" $v_0(r)$ defined by the equation

$$v_0(r) = \lambda_0 \int F(r) A(r, r'; \omega = 0) v_0(r') dr'. \qquad (5)$$

It is easy to see that in the limit under consideration $v_0(r)$ and $g(r)$ coincide, and $\lambda_0 = 1 + \alpha$ ($\alpha \ll 1$).

From (3) and (5) we easily obtain a formula for ω_0, generalizing the usual adiabatic formula derived for the case of multipole-multipole interaction

$$\omega_0^2 = \alpha \frac{(v_0 A(0) v_0)}{\left(v_0 \left(\frac{dA}{d\omega^2}\right)_0 v_0\right)}. \qquad (6)$$

For the ground state of a Fermi system, the change $\delta_0\rho$ of the quasiparticle-gas density with changing number of particles is of the same order in the interior and on the surface. Therefore the ratio of the overlap integrals $\kappa \sim (v_0\delta\rho)/(v_0\delta_0\rho)$ can be regarded as a certain characteristic of the state of the system; as already mentioned, $\kappa \sim 1$ for a gas and $\kappa \sim A^{1/3}$ for a liquid. Multiplying (2) from the left by $v_0(r)$ and using (5), we easily obtain

$$\kappa = \frac{1 + \alpha}{\alpha} \qquad (7)$$

i.e., at $\alpha \sim A^{-1/3}$ the system behaves already not like a gas but like a liquid. In this case, $\omega_0 \sim \epsilon_F A^{-1/3}$ (see (6)).

Let us determine under which conditions this takes place. In a system with large dimensions the propagator A can be calculated by the quasiclassical methods developed in [4]. We assume for simplicity that the self-consistent field U(r) does not depend on the quasiparticle energy and has a rectangular form. We neglect the dependence of F on the momenta, and assume in addition that $F = 0$ when $r < R$, and $F = F^{vac}$ when $r \geq R$. (Such a model, of course, is very crude, but the results are qualitatively the same also in more complicated cases.) We consider first the spherically symmetrical case (0^+ oscillations). It is then necessary to calculate only the integral $A_0(r, r') = \int A(\vec{r}, \vec{r}') d\vec{n}$. These calculations are rather lengthy, and we present only the result, accurate to some immaterial corrections:

$$A_0(r, r') = -\frac{2}{\pi^2}\frac{Mp_F^2}{R^2}\int_0^1 y^2(1 - y^2)^{1/2}\exp[-2yp_F(r_> - R)]\,dy. \qquad (8)$$

Introducing a new variable $x = p_F(r - R)$ and a new function $\psi(x) = \int_0^x v_0(x)$ we obtain, after substituting (8) in (5) and differentiating both halves of the equation with respectto x, the following differential equation for $\psi(x)$, with the conditions $\psi(0) = 0$ and $\psi(\infty) = \text{const}$:

$$\frac{d^2\psi}{dx^2} + U(x)\psi(x) = 0, \qquad (9)$$

where

$$U(x) = -4\lambda_0 f\int_0^1 y^3(1 - y^2)^{1/2}\exp(-2yx)\,dy.$$

This is the standard problem of finding the minimum depth of a three-dimensional spherically-symmetrical well, at which the first discrete level appears. A numerical solution yields $\lambda_0 f \simeq -6$, i.e., $\lambda_0 = 1$ at $f^{cr} = F^{vac}p_F^{cr}M/\pi^2 = -6$ (ρ_0^{cr} is the system density for which $\lambda_0 = 1$; $\rho_0^{cr} = (\rho_F^{cr})^3/3\pi^2$).

Let the ground state of a system with finite dimensions have a volume density $\rho_0 < \rho_0^{cr}$. Then the surface oscillations lie high, the integrals $(v_0\delta\rho)$ and $(v_0\delta_0\rho)$ are of the same order, the system behaves like a gas, and the added particles "land" in the volume, thereby increasing ρ_0. This continues until ρ_0 comes close to ρ_0^{cr}, so that the ratio $\kappa = (v_0\delta\rho)/(v_0\delta_0\rho)$ becomes of the order of $A^{1/3}$. Then, when particles are added, the volume density ρ_0 ceases to increase, and the main change $\delta\rho$ occurs on the surface of the system, i.e., the system behaves like a liquid in the vicinity of ρ_0^{cr}. The frequencies of the collective oscillations with $I \neq 0$ lie then alongside with $\omega_0 \sim \varepsilon_F A^{-1/3}$ and form a rotational band $\omega_I^2 = \omega_0^2 + \beta I(I + 1)/A$. Under certain conditions, apparently, the system can have a density $\rho_0 > \rho_0^{cr}$. Then the surface distribution of the particles should become modified in such a way that ω_0 does not vanish. One of the possible manifestations of such a modification is the occurrence of surface pairing of the particle-hole type. This problem will be considered separately.

In conclusion, the author is deeply grateful to A.B. Migdal, V.M. Galitskii, R.O. Zaitsev, V.M. Osadchiev, E.E. Sapershtein, M.A. Troitskii, and S.A. Fayans for a fruitful discussion.

[1] V.M. Osadtshiev and M.A. Troitsky, Phys. Lett. 26B, 421 (1968).
[2] L.D. Landau and E.M. Lifshitz, Mekhanika sploshnykh sred (Fluid Mechanics), GITTL, 1953 [Addison-Wesley, 1958].
[3] A.B. Migdal, Teoriya konechnykh fermi-sistem i svoistva atomnykh yader (Theory of Finite Fermi Systems and Properties of Atomic Nuclei), Nauka, 1965.
[4] W. Kohn and L. Sham, Phys. Rev. 137A, 1697 (1964).