Calculation of anomalous dimensionalities in non-Abelian field gauge theories

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We obtain the Gell-Mann–Low function and the anomalous dimensionalities in non-Abelian gauge theories with large numbers of fermions and bosons. The existence of a fixed point is proven.

Non-Abelian gauge theories were first discovered by Yang and Mills[1] and were for a long time as a mathematical toy, until it was recently observed that this theory has the remarkable property of asymptotic freedom.[2] Unlike all other renormalized theories,[3] the effective interaction parameter (invariant charge) $g^2(p^2)$ decreases with increasing momentum p.

If we fix the magnitude of the interaction at some Euclidean point $p^2 = \mathbf{p}^2 + \mathbf{p}_4^2 = \mu^2 > 0$, then $g^2(p^2)$ tends to zero at $p^2 \gg \mu^2$ like $[1 + a\ln(p^2/\mu^2)]^{-1}$. What is the physical meaning of the normalization point μ? It is usually stated that μ is the order of the hadron mass m, $g^2(\mu^2)$ is of the order of the physical charge, and the region $p^2 \gg \mu^2$ is the ultraviolet region that can be observed in deep inelastic reactions. We prefer a contrary point of view, based on the analogy with the theory of phase transitions.[4,5] We regard μ as the ultraviolet cutoff radius, which must be allowed to tend to infinity in the renormalized quantities, and $g^2(\mu^2)$ as the bare charge that is independent of μ. From our point of view, the region $p^2 \gg \mu^2$ is not observable, and the deep inelastic reactions correspond to the region $m^2 \ll p^2 \ll \mu^2 \to \infty$.

We start from the Lagrangian density

$$L = -\frac{1}{4}[\partial_\mu B_\nu^a - \partial_\nu B_\mu^a - ig_0 C_{abc} B_\mu^b B_\nu^c]^2 + \bar{\Psi}[i\gamma_\mu \partial_\mu - g_0 \sigma^a \gamma_\mu B_\mu^a]\Psi$$

$$+ \partial_\mu \Phi_a^* (\partial_\mu \Phi_a - ig_0 C_{abc} B_\mu^a \Phi_c) - \frac{1}{2a_0}(\partial_\mu B_\mu^a)^2, \quad (1)$$

where B_μ^a are vector gauge fields, Ψ are massless fermion fields, and Φ_a are the Faddeev-Popov ghosts.[6] C_{abc} are the structure constants of the Lie group G, and σ^a are the matrices of the fermion representation R of this group. g_0 is the bare charge and a_0 is the bare gauge. We use the Euclidean matrix $p^2 = (\mathrm{p}^2 + p_4^2)$ with $p_4 = ip_0$ real.

The equation of the renormalization group for the invariant charge $g^2(p^2)$ in the transverse gauge $a_0 = 0$ are[2,7]

$$\frac{dg^2}{d\ln p^2} = \beta(g^2). \quad (2)$$

The asymptotic behavior of the theory at $p^2 \ll \mu^2$ is determined by the fixed point of the renormalization group, i.e., by the zero $\beta = 0$ of the right-hand side of (2).

The first Taylor coefficient of this group was obtained by Gross, Wilczek, and Politzer[2]

$$\beta \to -ag^4/16\pi^2 + \ldots, \quad (3)$$

$$a = \frac{11}{3}C_2(G) - \frac{4}{3}T(R). \quad (4)$$

Here $C_2(G)$ is the Casimir operator of group G

$$\sum_{bc} C_{abc} C_{a'bc} = C_2(G)\delta_{aa'}, \quad (5)$$

and $T(R)$ is defined by

$$Tr(\sigma^a \sigma^{a'}) = T(R)\delta_{aa'} \quad (6)$$

We have calculated the following coefficient:

$$\beta = -\frac{ag^4}{16\pi^2} + bg^6/(16\pi^2)^2 + \ldots, \quad (7)$$

$$b = \frac{20}{3}T(R)C_2(G) + 4T(R)C_2(R) - \frac{183}{16}C_2^2(G). \quad (8)$$

Here $C_2(R)$ is the Casimir operator of the fermion representation R, i.e.,

$$\sum_a \sigma^a \sigma^{a'} = C_2(R)I. \quad (9)$$

The details of the calculation will be published in a subsequent article. We used the t'Hooft-Veltman method[8] of analytic regularization.

This suggests the natural idea of making a relatively small by varying the group G and the representation a. Then the remaining terms in (7) can be discarded and we obtain the fixed point

$$\frac{g_f^2}{16\pi^2} = \frac{a}{b(a=0)} + 0(a^2). \quad (10)$$

It is important that in (8) b is *positive* when $a = 0$

$$b(a=0) \equiv b_0 = \left(7 - \frac{5}{48}\right)C_2^2(G) + 11C_2(G)C_2(R) > 0. \quad (11)$$

This means that the fixed point (10) is infrared-stable. If we fix the charge $g^2(\mu^2)$ at $p^2 = \mu^2$ (bare charge), then the invariant charge tends to this fixed point as $\mu \to \infty$ and at fixed p^2.[9]

The situation here is the same as in the theory of phase transitions[4,5] and a-expansion, which, as we suppose, recalls Wilson's ϵ-expansion.[5]

The anomalous dimensionalities Δ_n of the composite tensor fields $0_{\mu_1 \ldots \mu_n}$, which determine the momentum dependence of the moments of the structure function $F_{p,n}(x, q^2)$ of the dep inelastic ep and en scattering,[10] e.g., for the difference

$$\int_0^1 dx\, x^n (F_p - F_n) = B^{(-)}(n)(\sqrt{q^2})^{n+2-\Delta_n^{(-)}} \quad (12)$$

can be expanded in (a/b)

$$\Delta_n^{(-)} = n + 2 + \gamma_n^{(-)}(g_f^2),$$

$$\gamma_n^{(-)}(g_f^2) = \frac{2aC_2(R)}{b_0}\left[1 - \frac{2}{n(n+1)} + 4\sum_{k=2}^{n}\frac{1}{k}\right] + 0(a^2). \quad (13)$$

we have used here the Gross and Wilczek calculations of the functions $\gamma^{(-)}(g^2)$.

In this article we consider only the massless theory, so that our results correspond to the ultraviolet limit $q^2 \gg m^2$ of the real theory with mass.

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