Connection between the N-soliton solution of the modified Korteweg-de Vries equation with the solution of the KdV equation

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The nonlinear Miura transformation, [1] which transforms the N-soliton solution of the modified Korteweg-de Vries equation into the N-soliton solution of the Korteweg-de Vries equation, is the consequence of a unitary transformation of the operator relations connected with these equations.

The Korteweg—de Vries (KdV) equation

$$u_t + 6uu_x \pm u_{xxx} = 0 \tag{1±}$$

and the modified Korteweg—de Vries (MKdV) equation

$$v_t + 6v^2v_x \pm v_{xxx} = 0 \tag{2±}$$

have infinite numbers of polynomial conservation laws and can be solved by the method of the inverse scattering problem (ISP).

A special place among this equation is occupied by the modified KdV equation (2–), the single-parameter soliton family of solutions of which

$$v = 1 - \frac{2\nu^2}{\left[1-(1-\nu^2)^{1/2}\right]\left[1+\dfrac{2(1-\nu^2)^{1/2}}{1-(1-\nu^2)^{1/2}}\operatorname{ch}^2\eta\right]}$$

$$= 1 - \nu\sum_{\sigma=\pm 1}\sigma \operatorname{th}(\eta+\sigma\Delta), \tag{3}$$

{where $\eta = \nu[x-(6-4\nu^2)t]$ and $\Delta = (1/4)\ln[(1+\nu)/(1-\nu)]$, with $0<\nu<1$} is related to the remaining equations of the KdV type that can be solved by the ISP method. Equation (2–) also has a solution of the shock-wave type

$$v = \operatorname{th}(x-2t). \tag{4}$$

1. To apply the ISP method to Eq. (2–), we rewrite it in the form

$$L_t = i[L, A] = i(LA - AL), \tag{5}$$

where

$$L = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} i\frac{\partial}{\partial x} + v\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$A = 4\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} i\frac{\partial^3}{\partial x^3} - 3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\left(i\frac{\partial}{\partial x}v^2 + v^2 i\frac{\partial}{\partial x}\right)$$

$$- 3\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\left(i\frac{\partial}{\partial x}v_x + v_x i\frac{\partial}{\partial x}\right).$$

Solving the direct and inverse spectral problem for the operator L, [3] we can obtain solutions of Eq. (2) in the form

$$v = 1 + \frac{\partial}{\partial x}\ln\frac{f_1}{f_2},$$
$$f_{1,2} = 1 + \sum_{n=1}^{N}\sum_{NC_n} a_{1,2}(i_1,\ldots,i_n)\exp[-2(\eta_{i_1},\ldots,\eta_{i_n})] \tag{6}$$

$\sum_{NC_n}$ denotes the sum of all possible combinations of n elements out of N, while to coefficients $a_{1,2}(i_1,\ldots,i_n)$ are given by the formulas

$$a_{1,2}(i_1,\ldots,i_n) = \prod_{k<l}^{n} a_{1,2}(i_k, i_l),$$

$$a_{1,2}(i_k, i_l) = (\nu_{i_k}-\nu_{i_l})^2(\nu_{i_k}+\nu_{i_l})^{-2}(1\pm\nu_{i_k})(1\pm\nu_{i_l}),$$

$$a_{1,2}(i_k) = 1\pm\nu_{i_k},$$

$$\eta_{i_k} = \nu_{i_k}[x-(6-4\nu_{i_k}^2)t] + \eta^0_{i_k},$$

where ν_{i_k} and $\eta^0_{i_k}$ are arbitrary numbers, with all the ν_{i_k} different.

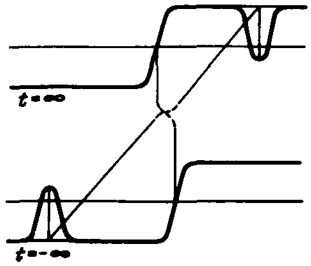
We carry out in (6) the similarity transformation $v\to\alpha^{-1}v$, $x\to\alpha x$, $t\to\alpha^3 t$. Then with $\alpha\nu_{i_k}$ finite as $\nu_{i_k}\to 0$, we go over to solutions $v = v+\alpha$ that vanish at $\pm\infty$, and expression (6) goes over into the exact N-soliton solution of the KdV equation. [2] A detailed discussion of this circumstance is contained in[4].

The solution (6) describes two essentially different cases.

a. If $\nu_k \neq 1$ $(k-1,\ldots,N)$, then we have a system of interacting solitons and the solution at $t\to\pm\infty$ can be represented in the form

$$v = 1 - \sum_{k=1}^{N}\sum_{\sigma=\pm 1}\sigma\nu \operatorname{th}(\eta_k+\sigma\Delta-\delta_k^{\pm}), \tag{7}$$

where δ_k^+ and δ_k^- are the phases of the solitons at $+\infty$ and $-\infty$, respectively. Thus, the solution breaks up at



Interaction of soliton with shock wave for the modified Korteweg—de Vries equation.

$t \to \pm\infty$ into N solitons, which collide pairwise with one another during their temporal evolution from $-\infty$ to $+\infty$, as a result of which the faster kth soliton acquires a positive phase shift (δ_{ki}) and the slower ith soliton acquires a negative shift ($-\delta_{ki}$)

$$\delta_{ki} = \delta^{+}_{ki} - \delta^{-}_{ki} = \ln\frac{\nu_i + \nu_k}{\nu_i - \nu_k} . \tag{8}$$

b. If the numbers ν_k include also $\nu_0 = 1$, then the solution at $t \to \pm\infty$ is a superposition of solitons and a shock wave

$$v = \mathrm{th}(\eta_0 - 2\delta^{\pm}_0) \pm \sum_{k=1}^{N} \sum_{\sigma=\pm 1} \sigma \nu_k \,\mathrm{th}(\eta_k + \sigma\Delta_k - \delta^{\pm}_k), \tag{9}$$

where the sign + (−) in front of the sum corresponds to t at $-\infty$ ($+\infty$). That is to say, the interaction of the soliton with the shock wave causes the soliton to topple and to acquire a positive phase shift (δ_{k0}), while the shock wave acquires a negative shift ($-2\delta_{k0}$) (see the figure)

$$\delta_{ko} = \frac{1}{2}\ln\frac{1+\nu_k}{1-\nu_k} , \tag{10}$$

Incidentally, the considered process can be interpreted as an interaction of three shock waves, and this is the cause of the doubled phase shift in the first term of the right-hand side of (9).

2. It is easily seen that Eq. (2) is equivalent also to the operator relation

$$(L^2)_t = i[L^2, A]. \tag{11}$$

We subject the operators L^2 and A to the unitary transformation $L^2 \to T^{-1}L^2T = \hat{L}^2$, $A \to T^{-1}AT = \hat{A}$, where T is a unitary operator

$$T = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{i\frac{\pi}{4}} & e^{-i\frac{3\pi}{4}} \\ e^{-i\frac{\pi}{4}} & e^{-i\frac{\pi}{4}} \end{pmatrix} \tag{12}$$

and it follows then from the operator equation

$$(\hat{L}^2)_t = i[\hat{L}^2, \hat{A}] \tag{13}$$

that

$$u = v^2 \pm v_x \tag{14$\pm$}$$

satisfies the KdV equation (1−). This result was discovered by Miura[1] from a comparison of the structures of the conservation laws of Eqs. (1) and (2). Substituting (6) in (14) and going over to solutions that vanish at infinity, we obtain an N-soliton family of solutions for Eqs. (1−), analogous to the usual family.[2]

Let us study first the temporal asymptotic forms of (6) and (14) in case a. As $t \to \pm\infty$ the solution u of Eq. (1−) breaks up into N solitons. Their form is determined from formula (14) [with v from (3)] and is given by

$$u = -2\nu_k^2 \,\mathrm{sech}^2(\xi_k \pm \Delta_k - \delta^{\pm}_k) \tag{15}$$

$\xi_k = \nu_k(x - 4\nu_k^2 t)$, where the positive (negative) sign preceding Δ_k corresponds to a positive (negative) sign in front of v_k. This means that in the course of the temporal evolution from $-\infty$ to $+\infty$ the solitons collide pairwise with one another without changing their shape, and acquire the phase shifts (8).

We shall show that the solution corresponding to case b also goes over with the aid of the transformation (14) into an N-soliton solution of the KdV equation.

The solution (14) [with v from (9)] breaks up at $t \to \pm\infty$ into $N+1$ solitons, where the soliton with the largest amplitude is produced by the shock wave. As $t \to -\infty$, the soliton shape is given by (15), where the sign of Δ_k is positive, while at $t \to +\infty$ the sign of Δ_k is negative. The different signs of Δ_k at $t \to \pm\infty$ are due to the toppling of the soliton upon interaction with the shock wave in the solution of the MKdV equation. The phase shift due to the interaction of a soliton with the maximal soliton is calculated from the formula

$$\delta_{ko} + 2\Delta_k = \ln\frac{1+\nu_k}{1-\nu_k} .$$

The maximal soliton acquires in this case a negative phase shift ($-2\delta_{k0}$). The solution (14+) [with v from (9)] breaks up at $t \to \pm\infty$ into N solitons. The signs of Δ_k are in this case the opposite of the signs in (14−), and this is the cause of the vanishing of the phase shift connected with the shock wave in the solution of the MKdV equation.

3. Using the solution (3) of Eq. (2−), we can easily obtain a single-parameter periodic solution of Eq. (2+). Carrying out in (3) and (2−) the transformation $x \to ix$, $t \to it$, we obtain

$$v = 1 - \frac{2\nu^2}{[1-(1-\nu^2)^{1/2}]\left[1 + \frac{2(1-\nu^2)^{1/2}}{1-(1-\nu^2)^{1/2}}\cos^2\eta\right]} \tag{16}$$

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[3]V.E. Zakharov, Zh. Eksp. Teor. Fiz. 64, 1627 (1973) [Sov. Phys.-JETP 37, No. 5 (1973)].
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