Slowed-down nonlinear damping of small-body oscillations at infralow temperature

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We consider the features of damping of mechanical vibrations of elastic bodies at infralow temperatures.

High-Q systems are of great interest for resonant registration of small periodic forces. The role of the quality factor Q was emphasized by Braginskii.[1]

It is known that when the temperature is lowered the damping decreases.[2,3] To describe the damping one uses the macroscopic concepts of thermal conductivity and internal friction (viscosity).

It follows therefore that if a definite oscillation mode is excited and its energy E is many times larger than the thermal energy, then the damping is exponential, dA/dt depends linearly on A, where A is the amplitude, and dE/dt depends linearly on E.

This answer, however, is based on definite assumptions that do not hold for small bodies and low temperatures. The damping of a given mode (frequency Ω) is a transfer of energy to other modes. Consequently, there occurs either spontaneous decay $\Omega \to \omega_i + \omega_k$, or the process $\Omega + \omega_i \to \omega_k$, in which the frequency of the thermal oscillations increases.

We consider for concreteness the lowest mode $\Omega = \min(\omega_n)$. Decay is then excluded and only the second process remains. We propose to consider a temperature such that $kT \ll \hbar\omega_i$ for all suitable pairs ω_i and ω_k satisfying the condition

$$\omega_k - \omega_i = \Omega.$$

If the condition $kT \ll \hbar\omega_i$ is satisfied, then the population of the ith mode is exponentially small, and accordingly the probability of dissipation of the mode via the given channel is small. In this case the dissipation will proceed in more complicated manners, such as $n\Omega + m\omega_i = l\omega_k$ or $n\Omega = q\omega_s + r\omega_t$, where n, m, l, q, and r are integers. Assume that we were able to satisy the foregoing condition with sufficiently small ω_i, such that $\hbar\omega_k \approx kT$. However, the process in which resonance calls for n quanta Ω leads to damping with a nonlinear and nonexponential law $dE/dt = -bE^n$, with an asymptotic form $E = [b(n-1)t]^{-(n-1)^{-1}}$ which is independent of E_0 in the limit of large t.

For numerical estimates it is necessary to know the concrete spectrum of the considered body, as well the width of the excited modes, and the latter is particularly difficult to determine.

For a body with volume V, at a sound velocity c, the asymptotic level density is given by $dN/d\omega \approx V\omega^2/c^3$. At a given Q of the higher levels α, i.e., at a level width $\Gamma = \omega/\alpha$, we obtain that value of ω at which we have $|\Omega + \omega_i - \omega_k| < \Gamma$ with a probability on the order of unity. We obtain

$$\int \Gamma \frac{dN}{d\omega} dN = \int \Gamma \left(\frac{dN}{d\omega}\right)^2 d\omega = 1, \qquad \frac{V\omega^3}{c^3\sqrt{\alpha}} = 1.$$

Consequently the mean-weighted statistical estimates gives for the first pair of levels for which a first-order process is possible numbers on the order of $N_i \approx N_k \approx \sqrt{\alpha}$ and corresponding frequencies $\omega_i \approx \omega_k \approx \Omega\alpha^{1/6}$, where α is the Q of the levels i and k which are capable of decaying spontaneously (unlike Ω). The estimate includes the large quantity α, so that $\omega_i \cong \omega_i \gg \Omega$. We can therefore observe the effect also if $\hbar\Omega < kT$. However, the 1/6 power is depressingly small. On the other hand, from the point of view described above, the damping should be particularly strong in the case when the frequency spectrum is equidistant, e.g., $\omega_{k+1} - \omega_k = \Omega$. Such a spectrum is obtained, for example, for longitudinal sound in a thin (r) rod bent into a ring of radius R (neglecting dispersion, $r \ll R$), In this case the generation of harmonics constitutes the formation of shock waves and the conversion of a sinusoidal wave into a sawtooth wave with subsequent damping. It is now clear why the loss of homogeneity over the length $2\pi R$ slows down the formation of the shock waves: the equidistant character of the frequency spectrum is destroyed. A certain enhancement of the effect is possible in the degenerate case of a symmetrical body in which there are selection rules for the interaction of the oscillations. For example, in the case of isotropic material in a spherical body, conservation of the angular momentum denotes (if Ω is radial) that the modes i and k should have equal $l_i = l_k$ and $m_i = m_k$. This limitation increases the power of α to 1/3 or 1/4 instead of 1/6. The power is also

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higher for an almost-one-dimensional thin body. However, in accord with the statements made above, it is important that the degeneracy, which limits the number of interacting levels, not lead to an equidistant behavior of the type $\omega_{i,l} = A_l + l_i$

To estimate the damping of the higher modes we can use the concepts of internal friction, kinematic viscosity, and thermal conductivity. In this case $\Gamma = A\omega^2$ and $\alpha \sim \omega$. Fixing ω_i by the quantization condition $\hbar\omega_i < kT$, we get $\Omega \approx \omega_i A^{1/6}$. We shall obtain a very rough numerical estimate based on the damping of oscillations in sapphire with $\alpha = 10^9$, as measured by Bagdasorov, Braginskii, and Mitrofanov[4] at $\omega \sim 10^5$ and $T = 7\,°K$. We extrapolate to $T = 10^{-3}$ in accord with the law $A \sim T^{-4}$. We choose the frequency $\omega_i = 10^9$ from the condition $\hbar\omega_i = 10kT$. We obtain $\alpha = \alpha_0(10^5/10^9)(7/10^{-3})^4 = 10^{20}$ (!) and $\Omega = \alpha^{-1/6}\omega_i = 2\times 10^6$. It must only be emphasized that under the exotic conditions considered above the damping time is quite large also in the internal-friction approximation ($\sim 10^9$ years), so that the observation of the effect calls for an exceptional experimental art.

We have formulated above a condition for the accuracy of the resonance. If this condition is not satisfied, $|\Omega + \omega_r - \omega_k| = \beta \gg \Gamma_r + \Gamma_k$, then the corresponding pair of levels makes no contribution to the damping of the fundamental mode Ω, but when the modes r and k are excited there occur beats of the amplitude of the fundamental mode with frequency β. At an observation time $\tau \lesssim \beta$, the additional difficulties lie in the fact that it is difficult to distinguish beats from damping or from the action of an external force. At $T = 10^{-3}\,°K$ and a sound velocity 10^6 cm/sec (sapphire) the condition $\hbar\Omega = kT$ for the fundamental tone would lead to the need of using a sphere of dimensions 3×10^{-2} cm, but since $\omega_{l,k} \gg \Omega$ it is not excluded that the described effect may be observable and body dimensions 0.5–1 cm, and perhaps even larger. The principal role is then assumed by elimination of the losses due to the suspension of the body and to electron and nuclear-spin excitations. The amplitudes of the oscillations may not be small, and therefore the quantum effects connected with observation of the oscillations can be overcome (at least in principle).

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