Homogeneous cosmological models with gravitational waves and rotation

V. N. Lukash

Moscow Physico-technical Institute
(Submitted February 25, 1974)
ZhETF Pis. Red. **19**, 499–503 (April 20, 1974)

Solutions are proposed for cosmological models with circularly polarized gravitational waves of arbitrary length λ and (or) vortical motions of matter of arbitrary scale **k**.

Exact solutions of the equations of general relativity theory for cosmological problems are of tremendous interest, particularly when it comes to explaining the relations between the present state of the universe and the character of the singularity. We consider below homogeneous cosmological models containing gravitational waves with circular polarization and (or) vortical motion of matter. Such an interpretation is admitted by certain models of type VII and VIII in accordance with the Bianchi classification (we present in this paper a

During the course of the evolution we trace the transition from the long-wave situation $\lambda > t$ to the wave situation, $\lambda < t$, where λ is the wavelength.[1)]

Owing to homogeneity, the exact solution of the equations of general relativity theory reduces to integration of several ordinary differential equations for the dependence of the parameters on the world time. The idea of the method is generalized to the case of electromagnetic waves, to simultaneous presence of waves (electromagnetic and gravitational) and matter having a definite equation of state, to vortical motion of matter, and to the case of the presence of viscosity, free (noninteracting) particles, and a plasma consisting of charge particles.

Ya. B. Zel'dovich has applied the idea of circularly-polarized waves to the problem of a strong electromagnetic wave in a rarefied plasma. Owing to the obvious symmetry (the electrons move in circles), the solution of the problem was reduced to algebraic equations.[1,2]

Let us turn to cosmology. The representation of the anisotropic model as an isotropic model on which a gravitational wave is superimposed has been under discussion for a long time. It has been usually assumed that in order for homogeneity to be exactly preserved the gravitational wave had to be infinitely long if it was superimposed on an infinite isotropic open or flat model. In the case of a closed model (the Friedmann closed world), the wavelength was assumed to be of the order of the world radius.[3-5][2)] This limitation was connected with the mental image of a wave with a periodic dependence of the parameters on the coordinates, with nodes and antinodes, which obviously disturbs the spatial homogeneity.

In the proposed class of cosmological solutions, circularly-polarized waves (standing and traveling) and vortical motions of matter are considered. The scalar quantities (the square of the amplitude, the curl, etc.) do not depend in this case on the coordinate z directed along the wave vector **k**. The vector and tensor quantities in the (x,y) plane perpendicular to the wave vector experience rotation when displaced along z. Thus, the translation group has a more complicated structure (a helical combination of a shift along z with rotation) than in the trivial case, but this does not exclude the construction of strictly homogeneous models with full equivalents of all the points of space (sections with $t = \mathrm{const}$).

In the period $\lambda < t$, the equations coincide with those obtained for the average quantities characterizing the large-scale metric, if the gravitational wave (and its pseudotensor) are considered on a par with any other type of wave, for example electromagnetic (see[10,11]). In this period ($t \to \infty$) the amplitude of the gravitational waves (μ) and the vortical velocity (β) are small, and the second-order perturbations are significant in the equations describing the evolution. In the general case, the amplitude and the curl increase when the singularity is approached. In the case of models with gravitational waves, it turns out that near the singularity, where the linear theroy developed by Lifshitz does not hold, its lutions, a particular quasiisotropic with an asymptotic singularity that differs little from the Friedmann solution (Fig. c), and a more general Kasner solution corresponding to an infinite perturbation (as $t \to 0$) of the Friedmann model (Fig. a). In addition, there exists also one type of singularity corresponding to infinite perturbation of the Friedman model, in which, however, the gravitation of matter is significant all the way to $t = 0$; see Fig. b (for details see[12,13]). In models with vortical motions of matter, none of the asymptotic forms is realized as $t \to 0$ for the solutions in which only gravitational waves are present (see the figure). In most general cases, the solution in the "vacuum" stage has an oscillatory character, similar to the well known oscillatory regime in the "mix master universe" model (type IX)[14,15] (models with gravitational waves and vortical motions of matter will be considered in detail separately).

We specify a metric in the form

$$ds^2 = dt^2 - g_{\alpha\beta}dx^\alpha dx^\beta; \quad x^1 \equiv x; \quad x^2 \equiv y; \quad x^3 \equiv z; \tag{1}$$

and with a tensor $g_{\alpha\beta}$

$$\begin{pmatrix} a^2(\mathrm{ch}\,\mu + \mathrm{sh}\,\mu \cos 2kz); & a^2\,\mathrm{sh}\,\mu \sin 2kz; & -a^2\nu e^{-\mu}\sin kz \\ a^2\,\mathrm{sh}\,\mu \sin 2kz; & a^2(\mathrm{ch}\,\mu - \mathrm{sh}\,\mu \cos 2kz); & a^2\nu e^{-\mu}\cos kz \\ -a^2\nu e^{-\mu}\sin kz; & a^2\nu e^{-\mu}\cos kz; & c^2 + a^2\nu^2 e^{-\mu} \end{pmatrix}, \tag{2}$$

where a, c, μ, and ν are unknown functions of the time t, the wave vector **k** (or 2**k** in the case of the gravitational wave) is directed along z and is an arbitrary constant parameter; $k \in (-\infty, +\infty)$.

The metric (2) is a particular case of a homogeneous cosmological model of type VII$_0$. In this metric, the following are the nonzero components of T^0_α (of the ener-

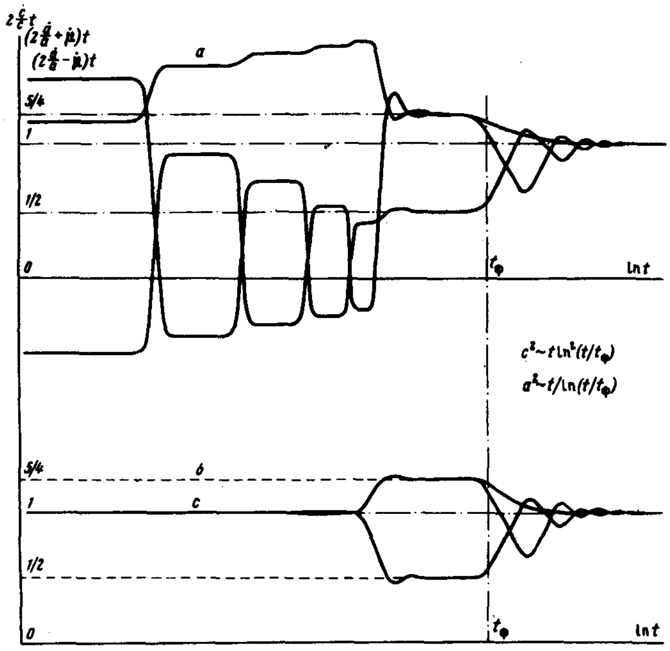

FIG. 1. Evolution of the principal values of the deformation tensor in the model with a gravitational wave ($\nu \equiv 0$); $p = \epsilon/3$.

gy-momentum tensor) describing the energy flux density:

$$T^{0}_{a} = (kR/a^{2}c)\{\cos kz;\ \sin kz;\ 0\};\quad R = -\frac{\dot{\nu}}{2}\,\frac{a^{4}}{c}\,e^{-\mu}; \tag{3}$$

where $a^2c = \sqrt{\det g_{\alpha\beta}}$; the dot denotes differentiation with respect to the synchronous time t. At $\nu \equiv 0$, the homogeneous space (1, 2) is co-moving, i.e., vortical motions of matter vanish. If furthermore $\mu \equiv 0$, then we are left with a diagonal metric with a flat co-moving space and there is no gravitational wave.

In conclusion, let us write out Einstein's equation for model (1), (2), filled with an equation of state $p = \alpha\epsilon$. (During the earlier stages of expansion $\alpha = 1/3$; later, when the role of radiation is small, $\alpha = 0$). The hydrodynamic equations can be integrated in finite form in this case and describe the conservation law for the momentum (absence of viscosity) and the energy (the velocity configuration is force-free—the vortex is collinear with the velocity {see (3)}:

$$kR = wa^{3}c\,e^{\mu/2}\,v\sqrt{1+v^{2}} = \text{const};\quad \epsilon^{\frac{1}{1+\alpha}}\,a^{2}c\sqrt{1+v^{2}} = \text{const}; \tag{4}$$

where $v = \beta/\sqrt{1-\beta^2}$, β is the three-dimensional velocity, $w = \epsilon + p$ is the enthalpy. The field equations take the form

$$\frac{\ddot{a}}{a} + \frac{\dot{a}}{a}\left(\frac{\dot{a}}{a} + \frac{\dot{c}}{c}\right) - \frac{R^{2}}{a^{6}}e^{\mu} = \frac{\epsilon - p}{2} + \frac{w}{2}v^{2}; \tag{5}$$

$$\frac{\ddot{c}}{c} + 2\frac{\dot{c}}{c}\frac{\dot{a}}{a} + \frac{2R^{2}}{a^{6}}e^{\mu} - \frac{2k^{2}}{c^{2}}\,\mathrm{sh}^{2}\mu = \frac{\epsilon - p}{2}; \tag{6}$$

$$\ddot{\mu} + \dot{\mu}\left(\frac{\dot{c}}{c} + 2\frac{\dot{a}}{a}\right) + \frac{2R^{2}}{a^{6}}e^{\mu} + \frac{2k^{2}}{c^{2}}\,\mathrm{sh}\,2\mu = wv^{2}; \tag{7}$$

$$\frac{\dot{a}}{a}\left(\frac{\dot{a}}{a} + 2\frac{\dot{c}}{c}\right) = \frac{R^{2}}{a^{6}}e^{\mu} + \frac{k^{2}}{c^{2}}\,\mathrm{sh}^{2}\mu + \frac{\dot{\mu}^{2}}{4} + \epsilon + wv^{2}. \tag{8}$$

Eqs. (5) and (6) describe the evolution of the metric; (7) is the wave equation determining the wave amplitude μ. The function $\nu(t)$ does not enter in the fundamental equations (5)—(8) and is determined independently from (3) and (4); $\nu \to 0$ as $t \to \infty$ (concerning the arbitrariness of $\nu(t)$ see [7]); Eq. (8) is the first integral of Eqs. (5)—(7).

In the model (1), (2) there are present only standing gravitational waves in the homogeneous space $t = \text{const}$. It is interesting that in open models of type VII (VII_h), traveling gravitational waves are possible in co-moving homogeneous space. (In particular, models with one traveling gravitational wave can be integrated in finite form).

The author thanks Ya. B. Zel'dovich for indicating many applications of the results described above, for help, and for numerous useful discussions. He is also grateful to I.D. Novikov and A.G. Doroshkevich for constant help and fruitful discussions, and to L.P. Grishchuk for numerous and useful discussions.

---

1) The speed of light and Einstein's constant are equal to unity.

2) L. P. Grishchuk formulated in most general form a hypothesis wherein any homogeneous model can be represented in the form of a more symmetrical model with perturbations (not necessarily small) that lower the symmetry. [6] We consider here those Bianchi VII models which are, according to Grishchuk, Friedmann worlds on which vector and tensor perturbations are superimposed (see [7-9].

---

1. Ya. B. Zel'dovich, in: Problemy Teoretichsko fiziki (Problems of Theoretical Physics), I. E. Tamm Memorial Volume, Nauka, 1972.
2. Ya. B. Zel'dovich and A. F. Illarionov, Zh. Eksp. Teor. Fiz. 61, 880 (1971) [Sov. Phys.-JETP 34, 467 (1972)].
3. L. P. Grishchuk, A. G. Doroshkevich, and V. M. Yudin, Abstracts of Papers, 3rd Soviet Gravitation Conference (in Russian), Erevan, 1972.
4. A. V. Byalko, Zh. Eksp. Teor. Fiz. 65, 849 (1973) [Sov. Phys.-JETP 38, 421 (1974)].
5. B. K. Berger, Univ. of Maryland Tech. Report 73-024, 1972.
6. L. P. Grishchuk, Proc. EGZIZU, Cracow, Poland, 1973.
7. E. M. Lifshitz, Zh. Eksp. Teor. Fiz. 16, 593 (1946).
8. M. Demianski, L. P. Grishchuk, Commun. Math. Phys. 25, 233 (1972).
9. L. P. Grishchuk, Bulletin de l'Academie Polonaise des Sciences 19, 12 (1972).
10. L. D. Landau and E. M. Lifshitz, Teoriya, Nauka, 1967, Classical Theory of Fields, Pergamon.
11. R. A. Isaacson, Phys. Rev. 166, 1263 (1968).
12. A. G. Doroshkevich, V. N. Lukash and I. D. Novikov, Zh. Eksp. Teor. Fiz. 64, 1457 (1973) [Sov. Phys.- JETP 37, No. 5 (1973)].
13. V. N. Lukash, Astron. Zh. 51, 281 (1974) [Sov. Astron.-AJ 18, No. 2 (1974)].
14. V. A. Belinskii, E. M. Lifshitz, and I. M. Khalatnikov, Usp. Fiz. Nauk 102, 463 (1970) [Sov. Phys.-Usp. 13, 745 (1971)].
15. A. G. Doroshkevich and I. D. Novikov, Astron. zhur. 47, 948 (1970) [Sov. Astron.-AJ 14, 673 (1971)].