Parametric excitation of second sound by first sound in liquid helium II

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We consider nonlinear combination (Raman) interaction of first sound with second sound in liquid helium II—parametric excitation of second sound by an exciting first-sound wave. An expression is obtained for the threshold intensity of the first sound, at which excitation the second-sound waves set in. The possibility of experimentally observing the phenomenon is discussed.

It is shown that propagation of first or second sound in liquid helium II can produce nonlinear phenomena due formally to nonlinear terms of the hydrodynamics equations. Osborne[1] observed the formation of second-sound shock waves, the theory of which was given in[2], and subsequently developed more accurately and in greater detail by Khalatnikov, [3] who considered also first-sound shock waves. The formation of shock waves is in essence self-action of waves and is not connected with nonlinear mixing of waves of different nature. In this paper we consider a different type of nonlinear interaction, namely nonlinear combination interaction of waves of first and second sound. It is well known that at sufficiently low intensities the ordinary and second sound propagate practically independently of each other. This is due, first, to the anomalously small coefficient of thermal expansion of liquid helium II, so that the two types of waves are not mixed in the linear approximation, and second, to the fact that the nonlinearity of the medium does not come into play at relatively wave intensities. At sufficiently high intensities, manifestations of nonlinear interactions can appear, one of which is the considered parametric excitation of second sound by first sound.

We start from the equations of the hydrodynamics of a superfluid liquid, [4] which can be transformed into

$$\left\{\begin{aligned}
&\frac{\partial^2 p}{\partial t^2} - c_1^2 \Delta p = c_1^2 \left\{ -\frac{\partial^2}{\partial t^2}\left[\frac{1}{2}\rho^2(\mathbf{V}_n - \mathbf{V}_s)^2 \frac{\partial}{\partial p}\frac{\rho_n}{\rho}\right] + \nabla\frac{\partial}{\partial x_k}(\rho_s v_{sk}\mathbf{v}_s + \rho_n v_{nk}\mathbf{v}_n)\right\};\\
&\frac{\partial^2 T}{\partial t^2} - c_2^2 \Delta T = -c_2^2 \frac{\rho_n}{\rho s}\frac{\partial s}{\partial t}\frac{\partial}{\partial t}\left(\frac{\rho}{s\rho_s}\right) - c_2^2\frac{\rho_n}{\rho s}\frac{\partial}{\partial t}\left\{\frac{1}{s\rho_s}\mathbf{V}_n\nabla(\rho s) + \frac{1}{s\rho_s}\frac{\partial}{\partial t}\left[\frac{1}{2}\rho(\mathbf{V}_n - \mathbf{V}_s)^2\left(\frac{\partial}{\partial T}\frac{\rho_n}{\rho} + \rho s\frac{\partial}{\partial p}\frac{\rho_n}{\rho}\right)\right.\right.\\
&\qquad \left.\left. - \frac{1}{\rho_s}(\mathbf{V}_s\nabla\rho_s + \mathbf{V}_n\nabla\rho_n) - \frac{1}{\rho_s}\frac{\partial}{\partial t}\left[\rho^2(\mathbf{V}_n - \mathbf{V}_s)^2\frac{\partial}{\partial p}\frac{\rho_n}{\rho}\right]\right]\right\}\\
&\qquad + c_2^2\frac{1}{\rho s}\left\{\nabla(\rho s)\nabla T + \nabla\rho_n\frac{\partial}{\partial t}(\mathbf{V}_n - \mathbf{V}_s) - \frac{1}{2}\nabla\left[\rho\nabla\left(V_s^2 - \frac{\rho_n}{\rho}(\mathbf{V}_n - \mathbf{V}_s)^2\right) + \frac{\partial}{\partial x_k}(\rho_s v_{sk}\mathbf{v}_s + \rho_n v_{nk}\mathbf{v}_n) + \mathbf{V}_s\frac{\partial\rho_s}{\partial t} + \mathbf{V}_n\frac{\partial\rho_n}{\partial t}\right]\right\}.
\end{aligned}\right. \tag{1}$$

Here p is the pressure, T the temperature, s the entropy, ρ the density, ρ_n and ρ_s the normal and superfluid densities, $\mathbf{V}_n$ and $\mathbf{V}_s$ the velocities of the normal and superfluid motions, and c_1 and c_2 the velocities of first and second sound. We assume that ordinary sound of high intensity propagates in the medium and excites parametrically, owing to the nonlinearity of the medium, two second-sound waves that draw energy from the exciting wave. To describe this process mathematically, we seek a solution of (1) in the form of a sum of three waves: an exciting first-sound wave $\frac{1}{2}[P\exp i(\mathbf{k}_0\cdot\mathbf{r}-\omega_0 t)+\text{c. c.}]$ and two excited second-sound waves $\frac{1}{2}[T_{1,2}\times\exp i(\mathbf{k}_{1,2}\cdot\mathbf{r}-\omega_{1,2}t)+\text{c. c.}]$ (T_2 is the backward wave). We assume here that the following equations, which correspond to the energy and momentum conservation laws, are satisfied:

$$\omega_0=\omega_1+\omega_2;\quad \mathbf{k}_0=\mathbf{k}_1+\mathbf{k}_2. \tag{2}$$

We assume that the waves propagate along one straight line. Since $c_1 \gg c_2$ (say at $T=1.5\,^\circ\text{K}$), it follows that (2) is satisfied under the condition

$$\omega_1\approx\omega_2\approx\frac{1}{2}\omega_0;\quad k_0=k_1-k_2.$$

Taking the damping of the waves into account, we obtain the following system of equations for the interacting-wave amplitudes that vary slowly with the distance:

$$\begin{cases}\dfrac{dT_1}{dx}+\dfrac{\alpha}{2}T_1-i\beta P T_2^*=0\\[2mm] \dfrac{dT_2}{dx}-\dfrac{\alpha}{2}T_2+i\beta P T_1^*=0\\[2mm] \dfrac{dP}{dx}+\dfrac{\gamma}{2}P-i\delta T_1T_2=0\end{cases}. \tag{3}$$

Here α and γ are the wave-amplitude damping factors,

$$\beta=\frac{\omega_1}{2c_2\rho_s}\left[\left(1+2\frac{\rho_s}{\rho}\right)\frac{1}{c_1^2}-\frac{\rho}{\rho_n}\frac{\partial\rho_n}{\partial p}\right];$$

$$\delta=\frac{c_1\omega_0\rho^2s^2}{2c_2^2\rho_n}\left[\left(1+2\frac{\rho_s}{\rho}\right)\frac{1}{c_1^2}-\frac{\rho}{\rho_n}\frac{\partial\rho_n}{\partial p}\right].$$

The boundary conditions are $T_1(0)=T_{10}$, $T_2(l)=0$, and $P(0)=P_0$, where l is the interaction length ($T_{10}\neq 0$ because of scattering by the fluctuations). From (3) we obtain the following expression for the threshold intensity of the exciting sound, at which excitation of temperature waves begins:

$$I_{\text{thr}}=\frac{1}{2\rho c_1}|P_{0\,\text{thr}}|^2=\frac{1}{2\rho c_1}\frac{\alpha^2}{4\beta^2}.$$

We obtain a numerical estimate for the threshold at $T=1.5\,^\circ\text{K}$. We assume

$$\omega_0=2\cdot 2\pi\cdot 10^5\ \text{sec}^{-1},\ \alpha=0.1\ \text{cm}^{-1},\ c_1=2.3\cdot 10^4\ \text{cm/sec};$$
$$c_2=2\cdot 10^3\ \text{cm/sec},$$
$$\rho=0.14\ \text{g/cm}^3,\ \rho_s=0.9\rho,\quad \rho_n=0.1\rho;$$

$\partial\rho_n/\partial p$ is estimated by using data on the pressure dependences of the velocity c_2, [5] of the entropy, and of the specific heat (see, e. g., [6]).

We obtain:

$$I_{\text{thr}}\approx 3\cdot 10^{-3}\ \text{W/cm}^2.$$

The solution of the system (3) (with damping disregarded)

$$\begin{aligned}T_1&=P_0\tilde{k}\sqrt{\frac{\beta}{\delta}}\,\text{sn}[K+\beta P_0(x-l),\tilde{k}],\\ T_2&=P_0\tilde{k}\sqrt{\frac{\beta}{\delta}}\,\text{cn}[K+\beta P_0(x-l),\tilde{k}],\\ P&=P_0\,\text{dn}[K+\beta P_0(x-l),\tilde{k}].\end{aligned} \tag{4}$$

Here sn, cn, and dn are elliptic functions, $K(\tilde{k})$ is a complete elliptic integral of the first kind, and $\tilde{k}$ is determined from the equation

$$\frac{T_{10}}{P_0}\sqrt{\frac{\delta}{\beta}}=\tilde{k}\,\text{sn}(K-l\beta P_0,\tilde{k}), \tag{5}$$

which in the case of sufficiently large amplification of the wave reduces to the form

$$l\beta P_0=K(\tilde{k}). \tag{6}$$

An investigation of the solution (4) and (6) shows that the intensity of the excited temperature waves becomes comparable with the initial intensity I_0 of first sound at a distance l on the order of several centimeters at $I_0=0.1\ \text{W/cm}^2$.

The described interaction can be used to obtain second sound of high frequency.

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