Analytic solution of the two-dimensional Korteweg–de Vries (KdV) equation

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The two-dimensional KdV equation is reduced to the form $\hat{L}_t = i[\hat{L}, \hat{A}]$ with the aid of a corresponding pair of linear operators $\hat{L}$ and $\hat{A}$. The question of the possibility of integrating the initial equation by the method of the inverse scattering-theory problem is discussed.

In[1,2], an equation was proposed for the description of the process of propagation of a two-dimensional perturbation in a nonlinear weakly-dispersing medium:

$$u_{tx} + (uu_x)_x + u_{xxxx} \pm \frac{1}{2} u_{yy} = 0. \quad (1)$$

This equation is a generalization of the well known one-dimensional Korteweg–de Vries (KdV) equation to the case when there is a weak dependence of the perturbation profile on the coordinate y which is transverse to its propagation. (The upper plus sign in (1) pertains to a medium with negative dispersion, and the lower minus sign to positive dispersion).

To trace the evolution of an arbitrary initial perturbation, with allowance for its non-one-dimensional character, it is necessary to investigate the solution of the Cauchy problem for Eq. (1).

In the present communication, we make use of the method of the inverse scattering-theory problem[3–6] to obtain such a solution. In accordance with the main idea of this method, we rewrite Eq. (1) in the form of an operator relation

$$\hat{L}_t = i[\hat{L}, \hat{A}] . \quad (2)$$

We choose as the operators $\hat{L}$ and $\hat{A}$ the following pair of operators

$$\hat{L} = 6\frac{\partial^2}{\partial x^2} - \sqrt{6}\frac{\partial}{\partial y} + u(x, y, t),$$
$$\hat{A} = -4i\frac{\partial^3}{\partial x^3} - iu\frac{\partial}{\partial x} - \frac{i}{2}u_x - \frac{i}{2\sqrt{6}}\int^x u_y\, dx \quad (3)$$

for a negatively dispersive medium and

$$\hat{L} = 6\frac{\partial^2}{\partial x^2} + i\sqrt{6}\frac{\partial}{\partial y} + u(x, y, t),$$
$$\hat{A} = -4i\frac{\partial^3}{\partial x^3} - iu\frac{\partial}{\partial x} - \frac{i}{2}u_x - \frac{1}{2\sqrt{6}}\int^x u_y\, dx \quad (4)$$

for a positively dispersive medium.

The solution of the Cauchy problem for Eq. (1) now reduces to a study of the direct and inverse spectral problems for the operators $\hat{L}$ from (3) and (4). In the case when the process of propagation of the perturbation is one-dimensional, Eq. (1) coincides with the well-known KdV equation, while the corresponding operator pair $\hat{L}$ and $\hat{A}$, with the aid of which this equation is integrated by the method of the inverse scattering-theory problem, is obtained from (3) and (4) by discarding the derivatives with respect to the coordinate y.

It follows from the form of the operators $\hat{L}$ [see (3) and (4)] that the direct and inverse spectrum problems for each of them turn out to be different, and this leads to different physical conclusions with respect to the singularities of the process of propagation of the perturbation in positively or negatively dispersive media. [1,2]

We note in conclusion that Eq. (1) can be obtained from a variation of principal with a Lagrange-function density in the following form:

$$L = \frac{\phi_t \phi_x}{2} + \frac{\phi_x^3}{6} + \frac{\psi^2}{2} + \phi_x \psi_x \pm \frac{\phi_y^2}{4},$$

where $\phi_x = u(x, y, t)$, $\phi_{xx} = \psi$, and $\phi_{tx} + \phi_x \phi_{xx} + \psi_{xx} \pm \phi_{yy}/2 = 0$.

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