Diagonality of the electron mass operators in a constant field

V. I. Ritus

P. N. Lebedev Physics Institute, USSR Academy of Sciences
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On the basis of arguments of general character, a proof is presented of the diagonality of the exact (outside the framework of perturbation theory with respect to external and radiation fields) mass operator of an electron in a constant electromagnetic field. The diagonality of the mass operator radically simplifies the study of radiation effects in an external field and reduces the integro-differential equations for the wave function and for the exact Green's function to algebraic equations.

In connection with the increasing interest in electrodynamics of intense field[1] and in calculations of the corresponding radiation effects,[2-7] a detailed proof of the diagonality of the exact mass operator of an electron in a constant electromagnetic field is presented in this article on the basis of the arguments given in[4,6], and the eigenfunctions that diagonalize this operator are written out.

As is well known, the motion of an electron in an external field, with allowance for the radiation corrections, is described by the modified Dirac equation proposed by Schwinger[8]

$$(i\gamma\pi + m)\phi(x) + \int M(x,x')\phi(x')d^4x' = 0, \quad \pi_\alpha = i\partial_\alpha - eA_\alpha. \tag{1}$$

The function $M(x,x')$, which describes the self-energy effects, can be regarded as the matrix element $M(x,x') = (x|M|x')$ of the mass operator M in the coordinate representation. The operator M is a scalar γ-matrix function of the operator π_α and the field $F_{\alpha\beta}$. For a constant field $F_{\alpha\beta}$ there exist only four independent scalars[6]

$$\gamma\pi, \quad \sigma F, \quad (F\pi)^2, \quad \gamma_5 FF^*, \tag{2}$$

on which the operator M can depend. All the remaining scalars are made up of these four, for example, $i\gamma F\pi = \frac{1}{4}[\gamma\pi, \sigma F]$, $i\gamma_5\gamma F^*\pi = \frac{1}{4}\{\gamma\pi, \sigma F\}$, $F^2 = \frac{1}{2}(\sigma F)^2 - \gamma_5 FF^*$, etc. It is easy to see that the operators (2), and with them the mass operator, commute with the squared Dirac operator $(\gamma\pi)^2 \equiv \pi^2 - \frac{1}{2}e\sigma F$, and therefore the operator M is diagonal in the representation of the eigenfunctions $E_p(x)$ of the operator $(\gamma\pi)^2$:

$$(\gamma\pi)^2 E_p = p^2 E_p \tag{3}$$

The eigenvalue p^2 of the operator $(\gamma\pi)^2$ can be any real number. Obviously, E_p is also an eigenfunction of the three other differential operators that commute with $\gamma\pi$ and whose eigenvalues number the solutions ψ of the usual Dirac equation

$$(i\gamma\pi + m)\psi = 0. \tag{4}$$

Thus, for a constant field of general form, the vectors **H** and **E** of which are directed along the axis 3 in a suitable coordinate frame, and the potential is chosen in the form $A = (0, Hx_1, -Et, 0)$, there exist four operators

$$(\gamma\pi)^2, \quad -i\partial_2, \quad -i\partial_3, \quad \pi_1^2 + \pi_2^2 - eH\Sigma_3, \tag{5}$$

which commute with one another and with the mass operator and form for the latter a complete system. The aggregate of the eigenvalues

$$p^2, \quad p_2, \quad p_3, \quad 2|eH|k, \qquad k = 0, 1, 2, \ldots \tag{6}$$

of the operators (5) will be denoted by the letter p. The operators (5) commute with Σ_3 and γ_5 [or with $\sigma F/2 = (H + iE\gamma_5)\Sigma_3$], and therefore the eigenfunctions differ also in the eigenvalues $s = \pm 1$ and $\gamma = \pm 1$ of the operators Σ_3 and γ_5.

In the so-called spinor representation (see[9], secs. 17–22 and[10], sec 8), in which γ_5 and Σ_3 are diagonal, the eigenfunctions $E_{ps\gamma}(x)$ take the form

$$E_{ps\gamma}(x) = \frac{e^{i\pi\lambda/4}\,\Gamma(-\lambda)}{(2\pi|E/H|)^{1/4}(n!)^{1/2}}\, e^{i(p_2x_2 + p_3x_3)} D_n(\rho) D_\lambda(\tau) w_{s\gamma}, \tag{7}$$

where $w_{s\gamma}$ are the proper bispinors of the matrices Σ_3 and γ_5; w_{1-1}, w_{-1-1}, w_{11}, and w_{-11} form in the considered representation columns 1, 2, 3, and 4 of a unit 4×4 matrix; $D_\nu(z)$ are parabolic-cylinder functions with indices

$$n = k + \frac{seH}{2|eH|} - \frac{1}{2}, \qquad \lambda = -i\,\frac{2|eH|k - p^2}{2|eE|} - \frac{\gamma s eE}{2|eE|} - \frac{1}{2},$$

and arguments

$$\rho = \sqrt{2|eH|}\left(x_1 - \frac{p_2}{eH}\right), \qquad \tau = e^{i\pi/4}\sqrt{2|eE|}\left(t + \frac{p_3}{eE}\right).$$

Thus, for a given p, the four bispinors (7) form a diagonal 4×4 matrix $E_p(x)$, made up of two diagonal 2×2 matrices ξ_p and η_p

$$E_p = \begin{pmatrix} \xi_p & 0 \\ 0 & \eta_p \end{pmatrix}, \quad [\pi^2 - p^2 - se(H - iE)]\xi_{ps} = [\pi^2 - p^2 - se(H + iE)]\eta_{ps} = 0, \tag{8}$$

the columns of which ξ_{ps}, η_{ps}, $s = \pm 1$ are two-component proper spinors of the Pauli matrix σ_3 and the solutions of the complex-conjugate equation (8). The spinors ξ_{ps} and η_{ps} transform independently in accordance with conjugate representations of the proper Lorentz group, and go over into one another upon reflection (the so-called four-spinors, see[9,10]).

The functions (7) are positive-frequency (as $E \to 0$): $E_p = E_{+p}$. The corresponding negative-frequency functions E_{-p} are obtained from them by the substitution $\tau \to -\tau$.

The matrix functions $E_p(x)$ are orthogonal and normalized in accordance with the condition ($\bar{E}_p \equiv \gamma_4 E_p^+ \gamma_4$, ω is the frequency sign):

$$\int \bar{E}_{\omega' p'}(x) E_{\omega p}(x)\, d^4x = (2\pi)^4 \delta(p'^2 - p^2)\delta(p_2' - p_2)\delta(p_3' - p_3) \times \delta_{k'k}\delta_{\omega'\omega}. \tag{9}$$

They satisfy the completeness condition:

$$\sum_{\pm,k} \int E_{\pm p}(x)\bar{E}_{\pm p}(y)\, \frac{dp^2 dp_2\, dp_3}{(2\pi)^4} = \delta(x - y). \tag{10}$$

The solution of the Dirac equation (4) is obtained by applying the operator $m - i\gamma\pi$ to the matrix $E_p(x)$, taken for $p^2 - m^2$, and coincides with Nikishov's solution.[11]

The obtained functions E_p correspond to fields for which both (or one) invariant of the field differs from zero. For field invariants equal to zero, these functions were obtained in[4,6]. The diagonality of the mass operator greatly simplifies the investigation of self-energy effects, and reduces the Schwinger integro-differential equations (1) to an algebraic equation. The exact Green's function of the electron is also expressed algebraically in term of the mass operator.

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