Collective quantum Hamiltonian in the model with variable moment of inertia

V. A. Khodel'

I. V. Kurchatov Institute of Atomic Energy
(Submitted June 10, 1974)
ZhETF Pis. Red. 20, No. 6, 419–423 (September 20, 1974)

A procedure is proposed for finding the parameters of the collective Hamiltonian. The methods developed are applied to an analysis of the model with variable moment of inertia, which describes with sufficient accuracy the properties of the fundamental excitation band of atomic nuclei in the transition region.

In the investigation of the collective motions of the nucleus, namely vibrations and rotations, extensive use is being made of phenomenological models based on the introduction of collective Hamiltonians with parameters taken as a rule directly from experiment. The foremost problem encountered in the microscopic calculation of these parameters is that of introducing collective variables that describe the considered class of excitations. Its solution is trivial at an interaction between particles $U(1,2)=\kappa(1)Q_\mu(2)$ [$Q(r)$ is an operator of any one of the multipole moments of the system]. In this case the collective parameter is Q itself, and the entire problem is solved in two stages[1,2]: first one finds the energy of the system at a fixed value Q, and then small oscillations relative to Q are taken into account. To construct $E(Q)$ one seeks the unconditional extremum of the functional $H'=H+\nu\hat{Q}$ and the Lagrange multiplier ν is defined in such a way that $\langle\hat{Q}\rangle=Q$. The term $\nu\hat{Q}$ can be regarded as a weak external field. We then have in the harmonic approximation

$$E(Q)-E(0)=-\frac{\nu^2}{2}P(\hat{Q},\omega=0)=-\frac{\nu^2}{2}(\hat{Q}G\mathcal{T}(Q)G), \tag{1}$$

$$Q-Q_0=\nu P(\hat{Q},\omega=0). \tag{2}$$

Here P is the polarization operator, G is the Green's function of the particle, and $\mathcal{T}$ is the vertex part, the

equation for which is[3]

$$\mathcal{T}(V_0) = V_0 + \mathcal{U} G \mathcal{T}(V_0) G, \tag{3}$$

where V_0 is the external field. Obviously, in the case when the irreducible interaction four-point diagram in the particle-hole channel is $\mathcal{U}(1,2,3,4) = \kappa Q(1) = \kappa Q(1) Q(2)\,\delta(1,3)\,\delta(2,4)$, the integral term in (3) is proportional to $\mathbf{Q}(r)$ regardless of the form of the external field $V_0(r)$. It is this which makes it possible to determine uniquely the collective parameter $\vec{\alpha} = \mathbf{Q}(r)$ in a system with multipole interaction.

It follows from (2) and (3) that

$$E(Q) - E(O) = C_Q \frac{\alpha^2}{2} + B_Q \frac{\dot{\alpha}^2}{2}, \tag{4}$$

$$C_Q = -P^{-1}(Q, \omega = 0), \quad B_Q = -\left(\frac{dP}{d\omega^2}\right)_{\omega=0} \Big/ P^2(Q, \omega = 0). \tag{5}$$

The validity of (5) can be easily verified using as an example the analytically-solvable problem of a single j-level.[1] Even in this model one can discern clearly the general principle used in the introduction of collective variables: it is necessary to introduce them in such a way as to describe the effective field completely by a minimum number of collective parameters. To simplify the manipulations we shall assume in the general case that the block $\mathcal{U}$ is potential: $\mathcal{U}(1,2,3,4) \equiv \mathcal{U}(1,2)\,\delta(1,3)\,\delta(2,4)$. Then the effective field $\mathcal{T}(r, r') \sim \delta(r - r')$ depends only on one argument, namely r, and it can be expanded in the eigenfields of the system, which are the eigenfunctions $\chi_n(r)$ of the kernel UGC[4]

$$\lambda_n \chi_n = UGG\chi_n. \tag{6}$$

The fields $\chi_n(r)$ do not change their form in the medium: if $V_0(r) = \chi_n(r)$, then in accordance with (3) and (6) we have $\mathcal{T}(\chi_n) = \chi_n(r)/(1 - \lambda_n)$. When $\mathcal{U}(1,2) \sim Q_\mu(1) Q_\mu(2)$ there is only one eigenfield $\vec{\chi}(r) = \mathbf{Q}(r)$, but in a real case their number is infinite. In a stable system all $\lambda_n \leq 1$. The value $\lambda_c = 1$ is critical. Any perturbation $V_0 \sim \chi_c(r)$ becomes infinitely amplified in the medium, and the system changes its state. If V_0 and χ_c are orthogonal, but other near-critical fields $\chi^0_{cr}(r)$, for which $(1 - \lambda^0_{cr}) \ll 1$ and $(V_0 \bar{\chi}^0_{cr}) = \nu^0_{cr}$, exist in the system, then $\mathcal{T}(r) = \nu^0_{cr} \chi^0_{cr}(r)/(1 - \lambda^0_{cr})$ + small increments. We see that in this case there is duplicated the situation of a system with multiple interaction forces—the effective field $\mathcal{T}(r)$ has a universal form $\mathcal{T}(r) \sim \chi^0_{cr}(r)$, and the external field contributes to the answer only the numerical parameter ν^0_{cr}. Then, in analogy with the preceding, we can introduce the collective-variable operator

$$a = \int \psi^+(r)\, \chi^0_{cr}(r)\, \psi(r)\, d^3 r. \tag{7}$$

Formulas (5) for C and B take the form

$$C_a = (1 - \lambda^0_{cr})/(\chi^0_{cr} \bar{\chi}^0_{cr}), \quad B_a = \left(\chi^0_{cr} \left(\frac{dK^0}{d\omega^2}\right)_{\omega=0} \chi^0_{cr}\right) \Big/ (\chi^0_{cr} \bar{\chi}^0_{cr})^2 \tag{8}$$

where we have introduced the notation $K^0(\omega) = \int G(\epsilon + \omega/2)\, G(\epsilon - \omega/2)\, d\epsilon / 2\pi i$ and $\chi^0_{cr} = K^0(\omega = 0)\, \chi^0_{cr}$. The oscillation frequency $\omega_s = (C/B)^{1/2}$ given by formula (8) coincides in the adiabatic limit ($\omega_s \ll \omega_{sp}$) with the frequency obtained by direct calculation from the equation for the transition amplitude.[4]

The near-critical eigenfields $\chi^0_{cr}(r)$ always exist in systems with spontaneous symmetry violation, when the Hamiltonian H commutes with a certain operator $\hat{R}$, while the mass operator Σ does not. The components of Σ that do not commute with $\hat{R}$ satisfy the following relation,[5,6] which is in essence a generalization of the Ward identity to include the case of symmetry violation

$$[\Sigma, R] = UK^0(\omega = 0)[\Sigma, R]. \tag{9}$$

It follows from (9) that in such systems the critical field is $\chi_c \equiv [\Sigma, R]$. Choosing for a liquid drop $\hat{R} = \hat{p}$, we obtain $\chi_c = \partial\Sigma(r)/\partial r$, and $\lambda_c = 1$ (the frequency of the dipole excitation of the center of gravity of the system is $\omega_1 \equiv 0$). It can be shown that owing to the condition (9) there exists not one but an entire spectrum of surface oscillations of a quantum drop, analogous to the ordinary capillary waves.[7] The methods of the theory of finite Fermi systems[8] make it possible to calculate with the aid of (8) and (9) the coefficients C_L and B_L for various mutlipolarities L.[7] It turns out that at $L \sim 1$ the quantities C_L and B_L differ significantly from the classical values customarily employed in the liquid-drop model. Thus, far from the closed shells, the quadrupole coefficient B_2 turns out to be much larger than the hydrodynamic coefficient $B_2^h = MA/2$, and the rigidity C_2, to the contrary, is smaller than C_2^h. (This behavior of C and B is due to quasiparticle transitions in the unfilled shells[9]). At a certain number of particles, C_2 can reverse sign, making it necessary to take into account the anharmonic terms:

$$E = E_0 + \frac{C_2}{2}\beta^2 + \frac{D\beta^4}{4} + B\dot{\beta}^2 + E\beta^3 \cos 3\gamma \cdots.$$

The coefficients D and E can be calculated on the basis of the Ward identity (9), in analogy with the determination of the scattering amplitude of a photon with small momentum K in qunatum electrodynamics with the aid of the usual Ward identity. The terms containing β to an odd power are as a rule small.[10] If it is recalled that $B_2/B_2^h > 1$ far from magic numbers, then we can advance the hypothesis that it is probably necessary to resort to the hydrodynamic Bohr model[11] for the description of many properties of nuclei in the transition region, since the main disparity between this model and experiment, namely the smallness of the hydrodynamic moment of inertia $J = 3B_2\beta^2$, disappears. If we assume also that the nuclei are axially symmetrical and that the mass coefficient B_2 depends little on β, then to find the spectrum of the principal rotational band it is necessary to minimize the functional

$$W(\beta) = \frac{C\beta^2}{2} + \frac{D\beta^4}{4} + I(I+1)/2J,$$

which can be easily transformed into

$$W = \frac{I(I+1)}{2J} + \frac{K}{2}(J - J_0)^2, \tag{10}$$

where

$$K = \frac{D}{18B_2^2} \qquad J_0 = -\frac{3C_2B_2}{D} \qquad J = 3B_2\beta^2. \tag{10'}$$

The Hamiltonian (10) is the Hamiltonian of the model with variable moment of inertia,[12,13] which describes successfully the spectrum of the fundamental excitation band of nuclei far from magic. It is seen from (10) that for spherical nuclei ($C_2 > 0$) the moment of inertia J_0 of the ground state is negative, and for deformed nuclei $J_0 > 0$ ($C_2 < 0$), as established in[13]. It follows from (10) that K increases quite rapidly as the magic numbers are approached (the mass parameter B_2 decreases rapidly). This tendency is confirmed by the experimental data (e.g., for isotopes of Ge and Os[12]). In the model with variable moment of inertia one obtains also a relation between the matrix element Q of the quadrupole transition between the ground and first 2^+ state, $Q_{02} = k[(J_0 + J_2)/2]^{1/2}$, where k is constant within 15—20% for all nuclei. A hydrodynamic approach based on formula (10) yields the close ratio $Q_{02} = 2J_2^2/(3J_2 - J_0)$. Of course, the essentially classical description with the aid of (10) is quite crude. Quantum corrections will be investigated in a separate paper.

In conclusion the author is deeply grateful to B. T. Geĭlikman, V. M. Galitskiĭ, D. P. Grechukhin, A. B. Migdal, G. A. Pik-Pichak, E. E. Sapershteĭn, M. A. Troitskiĭ, and S. A. Fayans for a discussion of the work.

[1]S. T. Belyaev, Zh. Eksp. Teor. Fiz. **39**, 1387 (1960) [Sov. Phys.-JETP **12**, 968 (1961)].
[2]A. Lane, Nuclear Theory Russ. transl. Nauka, 1967.
[3]A. A. Abrikosov, P. P. Gor'kov, and I. E. Dzyaloshinskiĭ, Primenenie metodov kvantovoĭ teorii v statisticheskoĭ fizike (Use of Quantum-Theoretical Methods in Statistical Physics), Nauka.
[4]V. A. Khodel', ZhETF Pis. Red. **16**, 410 (1972) [JETP Lett. **16**, 298 (1972)].
[5]H. I. Mikeska and W. Brenig, Z. Phys. **220**, 321 (1969).
[6]S. A. Fayans and V. A. Khodel', ZhETF Pis. Red. **17**, 633 (1973) [JETP Lett. **17**, 444 (1973)].
[7]V. A. Khodel', Yad. Fiz. **19**, 792 (1974) [Sov. J. Nucl. Phys. **19**, 404 (1974)].
[8]A. B. Migdal, Teoriya konechnykh fermi-sistem i svoĭstva atomnykh yader (Theory of Finite Fermi systems and Properties of Atomic Nuclei), Nauka 1965.
[9]S. T. Beljaev, Kgl. Dansk. Vid. Selsk Mat. fys Medd. **31**, 11 (1959).
[10]S. T. Belyaev and V. B. Zelevinskii, Zh. Eksp. Teor. Fiz. **42**, 1590 (1962) [Sov. Phys.-JETP **15**, 1104 (1962)].
[11]O. Bohr, Kgl. Dansk Vid Selsk Mat. Fys. Medd **26**, 14 (1952).
[12]M. Mariscotti, C. Sharf-Goldhaber, and B. Buck, Phys. Rev. **178**, 1864 (1969).
[13]A. Goldhaber and G. Sharf-Goldhaber, Phys. Rev. Lett. **24**, 1349 (1970).