Spectrum of states in the one-dimensional t'Hooft model

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We consider the theory of spinor and vector fields in two-dimensional space-time; this theory is invariant with respect to the internal symmetry group $SU(N)$. The interaction is of the Yang–Mills type. The model can be solved in the limit as $e^2 \sim N^{-1} \to 0$ (e is the coupling constant). The mass spectrum of the fermion-antifermion system, previously obtained by t'Hooft by summing a certain class of Feynman diagrams, is obtained.

The success of the quark picture has recently strengthened the interest in field-theory models in which the principal singularity is traced, namely, there are no physical states connected with the initial fermion field of the quarks, i.e., "the quarks do not fly out." In this connection, the one-dimensional models of Schwinger[1] and of t'Hooft[2] are worthy of interest, since they contain a mechanism that ensures the retention of the quarks. In the model of[2], in addition, an equidistant spectrum of the bound states is obtained.

In this article we describe an economical method of obtaining the particle spectrum in the model of[2]; this method makes it also possible to find a solution in the simpler model of[1]. We note that this method is of definite heuristic value, inasmuch as it is possible to investigate in its framework also other questions, particularly the scattering of bound states.

The model of[2] describes in one-dimensional space a Yang-Mills field $(A_\nu)_a^b$ that interacts with the quark field ψ^a. Here a and b are indices connected with the internal symmetry group $U(N)$. The Lagrangian takes the form

$$L = \bar{\psi}(i\,\partial_\nu \gamma^\nu - m)\psi - e\bar{\psi} A_\nu \gamma^\nu \psi - \frac{1}{4}\mathrm{Sp}\, G_{\mu\nu}G^{\mu\nu}, \qquad (1)$$

where

$$G_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ie[A_\mu, A_\nu]. \qquad (2)$$

The model of[2] corresponds to the limit $N \to \infty$ and to fixed $e^2 N$. Let us examine (1) in terms of variables on the light cone. It is convenient to use the notation $x^\mu \equiv (x_0, x^1) = 2^{-1/2}(t+z, -t+z)$, $x_\mu = \tilde{g}_{\mu\nu}x^\nu$, where $\tilde{g}_{\mu\nu} = -\left(\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right)$. If the symbol ∂_ν in (1) is taken to mean the derivative with respect to the usual coordinates t and z, then $\gamma^\nu = (\gamma^0, \gamma^1) = (\sigma_2, i\sigma_1)$, $\gamma_5 = \sigma_3$. The transition to the cone coordinates is effected by making the substitution $\gamma_\nu \partial_\nu \to \Gamma_\nu \partial/\partial x_\nu$, where

$$\Gamma^\nu = (\Gamma^0, \Gamma^1) = 2^{-1/2}(\gamma_0(1+\gamma_5), -\gamma_0(1-\gamma_5)).$$

Thus, in terms of the coordinates on the cone the Lagrangian again assumes the form (1), apart for the need for the substitution $\gamma_\nu \to \Gamma_\nu$ and for taking account of the fact that the dropping of the indices is effected by the tensor $\tilde{g}_{\mu,\nu}$. We agree to use for the coordinates the notation $(x^0, x^1) \equiv (\tau, \rho)$ and take τ to be the "time" and ρ the "coordinate."

Let $\psi^a \equiv \left(\begin{smallmatrix}u^a\\ v^a\end{smallmatrix}\right)$. We choose a gauge $A_1 = -A^0 = 0$. Then the independent dynamic variable is only the upper component u of the spinor ψ, since the derivatives of A^- and v with respect to the time τ do not enter in the Lagrangian. We eliminate A^1 and v with the aid of the equations of motion, and express the generators of the translations $\mathcal{P}^\nu = \int T^{0\nu} d\rho$ (where $T^{\mu\nu}$ is the momentum tensor) in terms of the independent variable $u^a(\rho, 0)$, subject to the boundary conditions

$$u^a(\pm\infty, \tau) = 0. \qquad (3)$$

The condition (3) is necessary for the existence of $\mathcal{P}^\nu$ and $\int L d\rho$. It turns out that (3) is compatible with the equations of motion and with translational invariance only if the following condition is satisfied:

$$Q_a^b = \int_{-\infty}^{\infty} J_a^b(\rho, 0)\, d\rho = 0, \qquad (4)$$

where

$$J_a^b = u_a u^b \equiv (u^a)^+ u^b. \qquad (5)$$

The quantities $Q_a{}^b$ are generators of the symmetry group $U(N)$. The equality (4) is "weak" in Dirac's terminology.[3] In quantum theory, Eq. (4) goes over into a condition on the states, so that only scalar representations of the $U(N)$ group are realized and, in particular, there are no free quarks. The condition (4) arises in classical theory as the consequence of the increase of the Coulomb forces with increasing distance. The presence of uncompensated charge in space would lead to a growth of the field at infinity and to the absence of translational invariance.

We put

$$u^a(\rho, 0) = 2^{-1/4} \sum_{\kappa > 0} (\alpha_\kappa^a e^{i\kappa\rho} + \beta_\kappa^{+a} e^{-i\kappa\rho}), \qquad (6)$$

and express $\mathcal{P}^\nu$ in terms of the variables α_κ and β_κ. The transition to the quantum theory is effected by imposing the usual anticommutation conditions on the quantities α_κ and β_κ, which are regarded as the quark and antiquark annihilation operators. These conditions stem from the requirement

$$\partial u^a(\rho, \tau)/\partial x^\nu = i g_{\nu\mu}[\mathcal{P}^\mu, u^a(\rho, \tau)]. \qquad (7)$$

In quantum theory the generators $\mathcal{P}^{\mu}$ take the form

$$\mathcal{P}^1 = -\sum_{\kappa>0}\left\{\frac{m^2}{2\kappa}\left(\alpha_\kappa^{+a}\alpha_\kappa^a + \beta_\kappa^{+a}\beta_\kappa^a\right) + \frac{e^2}{\kappa^2}\left[J_a^{+b}J_a^b + J_a^b J_a^{+b}\right]\right\}, \quad (8)$$

$$\mathcal{P}^0 = \sum_{\kappa>0}\kappa\left(\alpha_\kappa^{+a}\alpha_\kappa^a + \beta_\kappa^{+a}\beta_\kappa^a\right), \quad (9)$$

where the operators

$$J_b^a(\kappa) = 2^{-1/2}\sum_{q>0}\left[\alpha_q^{+b}\alpha_{q+\kappa}^a - \beta_q^{+a}\beta_{q+\kappa}^b + \theta(\kappa-q)\beta_{\kappa-q}^b\alpha_q^a\right] \quad (10)$$

result from the Fourier expansion of the current (5). The quantities $H \equiv -\mathcal{P}^1$ and $P \equiv \mathcal{P}^0$ can be naturally called the Hamiltonian and the momentum, respectively. As seen from (8) and (9), the state of the free particle with mass μ at a momentum eigenvalue κ has an "energy" $\mu^2/2\kappa$.

Let $H = H_0 + V$, where H_0 does not change the number of particles and is obtained from (8) by crossing out the terms that appear after structures of the type $\alpha\beta$ and $\alpha^+\beta^+$ are multiplied by the structures of the type $\alpha^+\alpha$ and $\beta^+\beta$. The neutral states of the type

$$|\psi(p)\rangle = \sum_{k=0}^{p}\phi(p, k)\,\alpha_k^{+a}\beta_{p-k}^{+a}|0\rangle \quad (11)$$

are included in the class of the states allowed by the condition (4) and possess the property

$$P|\psi(p)\rangle = p|\psi(p)\rangle, \quad (12)$$

$$H_0|\psi(p)\rangle = \epsilon(p)|\psi(p)\rangle, \quad (13)$$

with (13) being the eigenvalue equation. It is essential that $\epsilon(p)$ coincides with the exact energy, since it is a quantity of the order e^2N, whereas the corrections to this quantity arise in second order in V and have a scale $(e^2N)^2/N$. It follows from (11) and (13) that

$$\mu^2\left(\frac{e^2}{\pi}N\right)^{-1}F(p, x) = \left(1 + \frac{m^2}{e^2N}\right)\left(\frac{1}{1-x} + \frac{1}{x}\right)F(p, x) - \int_0^1 dy\,\frac{F(p, y) - F(p, x)}{(y-x)^2}, \quad (14)$$

where we use the notation $\epsilon(p) = \mu^2/2p$, $q = xp$, and $\phi(p, xp) = F(p, x)$. This equation was obtained in[2] by summing a definite class of Feynman diagrams. At $\mu^2 \gg e^2N$, its normalized solutions are obtained in explicit form by using the boundary conditions $F(p, 0) = F(p, 1) = 0$, which must be imposed in order for the Hamiltonian to be Hermitian. The wave function turns out to be

$$F_k(p, x) = \sqrt{\frac{8\pi}{pN}}\sin k\pi x, \quad k \text{ is an integer} \quad k \gg 1. \quad (15)$$

The mass spectrum is given by

$$\mu_k^2 = \frac{e^2N}{\pi}k\pi^2 \quad (16)$$

In principle, the effect of those terms of the Hamiltonian (8) which describe the production of fermion pairs can be taken into account with the aid of ordinary perturbation theory.

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