Graviton production in a Friedmann universe

B. V. Vaĭner and P. D. Nasel'skiĭ

Rostov State University

(Submitted November 3, 1975)
Pis'ma Zh. Eksp. Teor. Fiz. **23**, No. 3, 141–145 (5 February 1976)

It is shown that gravitons can be produced near a singularity if account is taken of the reaction of the energy of the long-wave potential excitations and vortical excitations on the course of the cosmological expansion in the homogeneous isotropic universe model.

PACS numbers: 04.70.+t

The production of gravitons and other particles near singularities was investigated in[1–3]. It was shown there that no gravitons are produced in a homogeneous isotropic model of a universe filled with ultrarelativistic matter with an equation of state $p=\epsilon/3$. In such a system, however, excitations of the scalar, vector, and tensor type can take place, and their energy alters the course of the cosomological expansion.[4] Indeed, the gravitation-theory equations that take into account the influence of the excitation energy on the behavior of the background quantities smoothed out over the averaging volume $V^{(3)}$ are of the following form:

$$\bar{R}_i^p < G_p^k > -\frac{1}{2}\delta_i^k \bar{R}_e^m < G_m^e> = \kappa \bar{T}_i^p < G_p^k> + \kappa < G_p^k \tau_i^p> - <\mathcal{P}_i^p G_p^k> + \frac{1}{2}\delta_i^k <\mathcal{P}_e^m G_m^e>, \tag{1}$$

$$\mathcal{P}_i^p G_p^k - <\mathcal{P}_i^p G_p^k> + \bar{R}_i^p (G_p^k - < G_p^k>) - \frac{1}{2}\delta_i^k \{\mathcal{P}_e^m G_m^e - <\mathcal{P}_e^m G_m^e>$$

$$+ \bar{R}_e^m (G_m^e - < G_m^e>)\} = \kappa\{G_p^k \tau_i^p - < G_p^k \tau_i^p> + \bar{T}_i^p (G_p^k - < G_p^k>)\}, \tag{2}$$

$$\mathcal{P}_i^k = \frac{1}{2} G_s^e (h_i^{s;k}{}_{;e} + h^{sk}{}_{;i,e} - h_i^{k;s}{}_{;e} - h_{e;i}^{s}{}^{;k}) + \frac{1}{4} G_p^e G_s^m \{(h_{e;m}^p - 2h_{m;e}^p)$$

$$\times (h_i^{s;k} + h^{sk}{}_{;i} - h_i^{k;s}) + (h_m^{p;k} + h_m^{k;p} - h^{kp}{}_{;m})(h_{i;e}^s + h_{e;i}^s - h_{ie}{}^{;s})\}. \tag{3}$$

We note that in the derivation of (1)—(3) we used the relations

$$\mathcal{G}_{ik}(x^e) = \bar{\mathcal{G}}_{ik}(x^0) + h_{ik}(x^e),$$

$$G_i^p(\delta_k^i + h_k^i) = \delta_k^p,$$

$$\bar{T}_i^e = (\bar{p} + \bar{\epsilon})\bar{u}_i \bar{u}^e - \delta_i^e \bar{p}, \tag{4}$$

$$\tau_i^e = (\bar{p} + \bar{\epsilon})(\bar{u}_i \delta u^2 + \bar{u}^e \delta \bar{u}_i + \delta u_i \delta u^e) + (\delta p + \delta\epsilon)(\bar{u}_i \bar{u}^e + \delta u_i \delta u^e$$

$$+ \bar{u}_i \delta u^e + \bar{u}^e \delta u_i) - \delta_i^e \delta p - h_i^e \bar{p} - h_i^e \delta p \qquad (i, k, e = 0,1,2,3),$$

where $\bar{p}$ and $\bar{\epsilon}$ are the pressure and energy density of the matter ($\bar{p}=\bar{\epsilon}/3$),

$$\epsilon_{tot} = \bar{\epsilon} + \delta\epsilon, \quad p_{tot} = \bar{p} + \delta p$$

$\bar{R}_i^k$ is the Ricci tensor defined on $\bar{\mathcal{G}}_{ik}(x^0)$.

In contrast to[4], the system (1)—(3) was derived from the Einstein equations

$$R_{ik} - \frac{1}{2}\mathcal{G}_{ik}R = \kappa T_{ik}$$

by averaging over the 3-volume $V^{(3)}$, the averaging operation being defined in the form

$$< f > = \frac{1}{\sqrt{-\gamma}\, V^{(3)}} \int f(\bar{x}, x^0)\sqrt{-\gamma}\, dV^{(3)}; \quad \gamma = \det \|\mathcal{G}_{\alpha\beta}\|; \alpha,\beta = 1,2,3 \tag{5}$$

with preservation of the requirement $\langle\delta\epsilon\rangle = \langle\delta p\rangle = \langle\delta u_i\rangle = \langle h_{ik}\rangle = 0$. Naturally, an averaged description of the system is possible if

$$\lambda_T^2 << \frac{1}{\bar{R}^{(3)}},$$

where λ_T is the characteristic wavelength of the excitations and $\bar{R}^{(3)}$ is the curvature of the three-dimensional space. For a homogeneous isotropic cosmological model, the interval of which is represented in the form $ds^2=a^2(\xi)\times(d\xi^2-dx^2-dy^2-dz^2)$ we present the solution of (1)—(3) for the scale factor $a(\xi)$

$$a^2(\xi) = R_0^2(\xi^2 - c_T^2),$$

where R_0^2 is a parameter of the model (10^{50} cm^2) and c_T=const is determined by specifying the amplitude of the corresponding sort of excitations (potential, vortical). It is seen from (6) that the 4-dimensional variant of the curvature $\bar{R}$, which vanishes identically in the dynamic case, is now determined by the level c_T^2 of the potential or vortical "turbulence,"

$$\bar{R} = -\frac{6c_T^2}{R_0^2(\xi^2 - c_T^2)^3}. \tag{6}$$

Using the Lifshitz classification[5] for tensor perturbations of the metric, we consider their evolution against a background distorted by other sorts of excitations. To this end we represent h_α^β in the form

$$h_\alpha^\beta(x^e) = G_\alpha^\beta \frac{f_k(\xi)}{a(\xi)} e^{i\mathbf{kr}} + \text{c.c.},$$

where $f_k(\xi)$ is the solution of the equation

$$f_k'' + \left(k^2 - \frac{a^2\bar{R}}{6}\right)f_k = 0; \quad f_k' = \frac{d}{d\xi} f_k. \tag{7}$$

In the case when $c_T=0$, i.e., in a universe filled with matter with $p=\epsilon/3$, Eq. (7) describes the propagation of adiabatic gravitons, and in accord with[2] no gravitons are produced. However, at c_T different from zero the number of gravitons is no longer an adiabatic invariant and one can consider a graviton production process in analogy with the procedure proposed in[1].

Indeed, the solution of (7) should be sought in the form

$$f_{\mathbf{k}}(\xi) = \frac{\alpha_{\mathbf{k}}(\xi)}{\sqrt{2\omega}} e^{-i\omega\xi} + \frac{\beta_{\mathbf{k}}(\xi)}{\sqrt{2\omega}} e^{i\omega\xi}, \quad \omega = |\mathbf{k}|.$$

Taking into account the additional condition on α_k and β_k

$$f'_{\mathbf{k}}(\xi) = -i\omega\left(\frac{\alpha_{\mathbf{k}}(\xi)}{\sqrt{2\omega}} e^{-i\omega\xi} - \frac{\beta_{\mathbf{k}}(\xi)}{\sqrt{2\omega}} e^{i\omega\xi}\right)$$

the final system of equations takes the form

$$\alpha'_{\mathbf{k}}(\xi) = \frac{i}{2\omega} \frac{c_T^2}{(\xi^2 - c_T^2)^2}\left(e^{2i\omega\xi}\beta_{\mathbf{k}}(\xi) + \alpha_{\mathbf{k}}(\xi)\right),$$

$$\beta'_{\mathbf{k}}(\xi) = -\frac{i}{2\omega} \frac{c_T^2}{(\xi^2 - c_T^2)^2}\left(e^{-2i\omega\xi}\alpha_{\mathbf{k}}(\xi) + \beta_{\mathbf{k}}(\xi)\right).$$

It has an exact integral

$$|\alpha_{\mathbf{k}}(\xi)|^2 - |\beta_{\mathbf{k}}(\xi)|^2 = \text{const}.$$

Assuming $\xi = \xi_0$, $\alpha_k = 1$, and $\beta_k = 1$ at the initial instant of time, and solving it by the method of successive approximations, we obtain

$$\beta_{\mathbf{k}}(\xi) = -\frac{i}{2\omega} c_T^2 \int_{\xi_0}^{\xi} \frac{e^{-2i\omega\xi}}{(\xi^2 - c_T^2)^2} d\xi. \tag{8}$$

The obtained solution, according to[1], describes graviton production in a varying metric. It is seen that the main contribution to (8) is made by the region $\xi \approx c_T$, and $\xi \gg c_T$ the production effect becomes negligibly small. Inasmuch as graviton production is analogous in a certain sense to the process of particle production in an alternating electromagnetic field, we reduce (7) to the form customarily used in quantum electrodynamics:

$$f_{\mathbf{k}}(\xi) = f_{\mathbf{k}}^{(0)}(\xi) - \frac{1}{6}\int_{\xi_0}^{\xi} G_{\mathbf{k}}(\xi - \xi')\bar{R}(\xi')a^2(\xi')f_{\mathbf{k}}(\xi')d\xi', \tag{9}$$

where $G_k(\xi - \xi')$ is the Green's function of (7) at $\bar{R} = 0$. Solving (9) by perturbation theory and confining ourselves to terms of first order in c_T^2, we obtain

$$f_{\mathbf{k}}(\xi) = \frac{1}{\sqrt{2k}} c_{\mathbf{k}} e^{-ik\xi} + \frac{1}{\sqrt{2k}} c_{\mathbf{k}}^{+} e^{ik\xi} + \frac{c_T^2}{4ik}\int_{\xi_0}^{\xi} \frac{c_{\mathbf{k}}^{+} e^{-ik\xi + 2ik\xi'} - c_{\mathbf{k}} e^{ik\xi - 2ik\xi'}}{(\xi'^2 - c_T^2)^2} d\xi'. \tag{10}$$

It follows from (10) that whereas there is no backward wave at the initial instant $\xi = \xi_0$, such a wave is produced at the subsequent instants $\xi > \xi_0$. Using the fact that the main contribution is made to the region $\xi \approx c_T$, the solution can be represented in the form

$$f_{\mathbf{k}}(\xi) = \frac{1}{\sqrt{2k}}\left(c_{\mathbf{k}} + \frac{c_{\mathbf{k}}^{+}}{4ik} I(k)\right)e^{-ik\xi} + \frac{1}{\sqrt{2k}}\left(c_{\mathbf{k}}^{+} - \frac{c_{\mathbf{k}}}{4ik} I(-k)\right)e^{ik\xi},$$

where

$$I(k) = c_T^2 \int_{\xi_0}^{\infty} \frac{e^{-2ik\xi'}}{(\xi'^2 - c_T^2)^2} d\xi'.$$

Defining in the usual manner the particle-number density in the Kth state and using the commutation relations for the bosons

$$[c_{\mathbf{k}} c_{\mathbf{k}}^{+}]_{-} = 1$$

we obtain

$$n_{\mathbf{k}}(\infty) = n_{\mathbf{k}}(\xi_0) + \frac{I^2(k)}{16k^2}(1 + n_{\mathbf{k}}(\xi_0)). \tag{11}$$

The last term in (11) describes graviton production on a distorted metric. It must be noted that the energy of the gravitons produced after a time $\xi \approx c_T$ can greatly exceed the perturbation energy, and the distortion of the evolution of the expansion can be neglected. Naturally, graviton production ceases in this case. It can be assumed, however, that graviton production will lead to generation of potential and vortical perturbations, which in turn will influence the course of the expansion.

The authors are grateful to L. S. Marochnik, N. V. Pelikhov, and A. M. Krimskiĭ for a useful discussion of the work.

[1]Ya. B. Zel'dovich and A. A. Starobinskiĭ, Zh. Eksp. Teor. Fiz. **61**, 2161 (1971) [Sov. Phys.-JETP **34**, 1159 (1972)].
[2]L. P. Grishchuk, Zh. Eksp. Teor. Fiz. **67**, 825 (1974) [Sov. Phys.-JETP **40**, 409 (1975)].
[3]Ya. B. Zel'dovich and I. D. Novikov, Stroenie i évolyutsiya Vselennoĭ (Structure and Evolution of Universe), Nauka, 1974, p. 704.
[4]L. S. Marochnik, N. V. Pelikhov, and G. M. Vereshkov, Astrophys. Space Sci. **34**, 233 (1975).
[5]E. M. Lifshitz, Zh. Eksp. Teor. Fiz. **16**, 587 (1946).