Nonlinear instability of plasma pinches

N. M. Zueva, L. S. Solov'ev, and A. I. Morozov

I. V. Kurchatov Institute of Atomic Energy

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We demonstrate the existence of a nonlinear MHD instability that leads to an appreciable modification of the internal structure of the magnetic field.

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The linear theory of the stability of plasma configurations has by now been developed in detail. It is known, however, ever since Lyapunov published his work on stability of gravitating masses that the linear approximation is sufficient only in those cases when the frequency ω is a complex quantity. If ω is real (the so-called "neutral case"), then the linear approximation is insufficient for the solution of the question of the stability at an arbitrarily small initial perturbation.[1] The problem of stability of equilibrium plasma configurations in the case of ideal conductivity pertains precisely to this "neutral case." Nonetheless, this was not discussed in the theory of the stability of plasma configurations. The nonlinear stages of evolution of a perturbed plasma pinch are of natural interest (see, e.g., [2,3]). In this connection, the present authors, jointly with Gerlakh, carried out a numerical computer calculation[4–6] of the evolution of straight plasma pinches under the influence of an initial velocity perturbation

$$v_{r0} = \lambda r^{m-1}(1 - r^2)\cos m\theta, \quad v_{\phi 0} = -\lambda r^{m-1}\left(1 - \frac{m+2}{m} r^2\right)\sin m\theta, \tag{1}$$

satisfying the condition $\mathrm{div}\,\mathbf{v}_0 = 0$. Here $\theta = \phi - \alpha z$, $\alpha = 2\pi/L$, $m = 2$, and $v_{z0} = 0$.

We calculated the system of MHD equations of an ideal plasma

$$\frac{\partial \rho}{\partial t} + \mathrm{div}\,\rho\mathbf{v} = 0, \qquad \frac{\partial \mathbf{B}}{\partial t} = \mathrm{rot}\,[\mathbf{v}\times\mathbf{B}],$$
$$\frac{\partial s}{\partial t} + \mathbf{v}\nabla s = 0, \qquad \rho\frac{d\mathbf{v}}{dt} = -\nabla p + [\mathbf{j}\times\mathbf{B}], \tag{2}$$

where $\mathbf{j} = \mathrm{curl}\mathbf{B}$ and $p = \rho^\gamma \exp s$, under the assumption of helical symmetry of the process (the independent variables are r, θ, and t). The initial equilibrium configuration was specified in terms of the dimensionless parameters

$$B_{z0} = 1, \qquad j_{z0} = 2\mu_0(1 - r^{2N}), \qquad N = 1; 10. \tag{3}$$

The case $N = 1$ corresponds to a parabolic distribution of the current, and the case $N = 10$ to a quasihomogeneous one. In the course of the calculation it became clear that at definite "resonant" parameters μ_0/a all the considered configurations "crumbled" after several units of time (Figs. 1a and 1b), although the amplitudes of the velocity perturbation (measured in Alfven units) were chosen to be small enough ($\lambda = 10^{-1}$; 10^{-2}).

According to classical linear theory of stability, the considered configura-

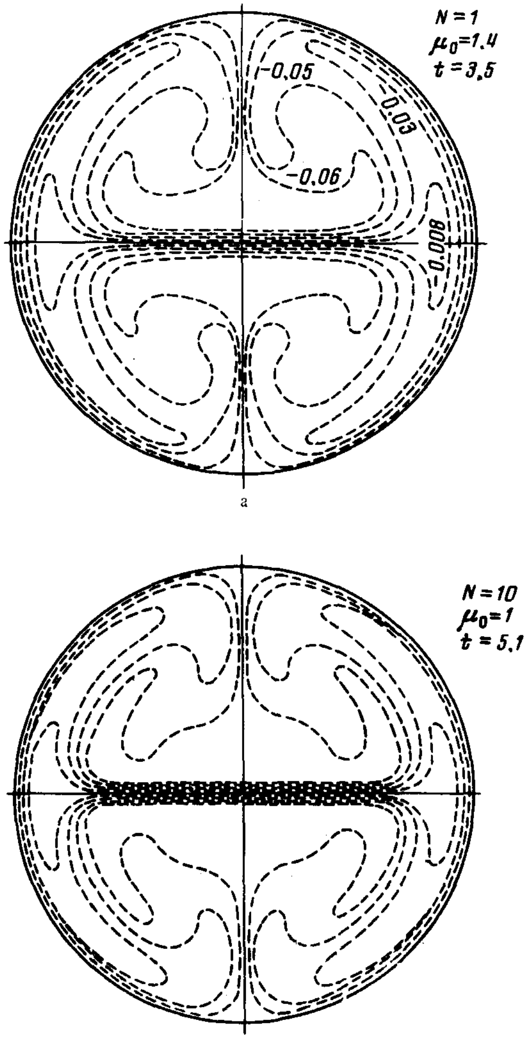

FIG. 1. Sections through the magnetic surfaces: a) $N=1$; $\mu_0/\alpha=1.4$; $t=3.5$. b) $N=10$; $\mu_0/\alpha=1$; $t=5.1$.

tions are stable at the given perturbations, since the potential-energy increment[7] $\delta W>0$ (see Fig. 2).

To understand the observed strange phenomena, we seek the solution of the system (1) by the linearization method, wherein the nonlinear terms are discarded and, following Trubnikov,[8] by the method of expanding all parameters in powers of time t

$$a=a_0+a_1t+\frac{a_2}{2!}t^2+\cdots.$$

We then obtain the following expression for the time variation of the kinetic energy of the plasma configuration

$$W_k = \frac{1}{2}\int \rho_0 v_0^2\, d\tau + \frac{t^2}{2}\int \rho_0 v_0 v_2\, d\tau + \frac{t^4}{6}\int (\rho_0 v_2^2 + 0(v_0^4))\, d\tau + \cdots$$

$$= W_{k0} - t^2\lambda^2 A_2 + t^4\lambda^2 A_4 + 0(t^4\lambda^4) + \cdots . \qquad (4)$$

where

$$\rho_0 \mathbf{v}_2 = \nabla(\mathbf{v}_0 \nabla p_0 + \gamma p_0 \operatorname{div} \mathbf{v}_0) + [\mathbf{j}_1 \times \mathbf{B}_0] + [\mathbf{j}_0 \times \mathbf{B}_1], \quad \mathbf{B}_1 = \operatorname{rot}[\mathbf{v}_0 \times \mathbf{B}_0].$$

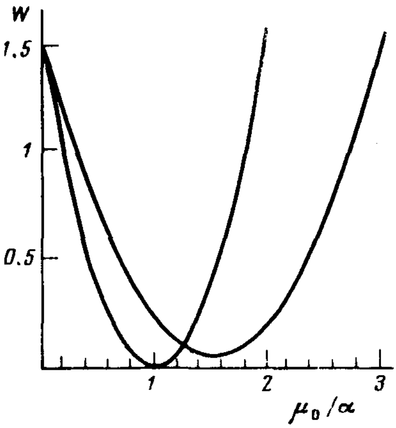

FIG. 2. Potential energy of the approximation $N=1$ (a) and $N \gg 1$ (b).

The expression for the second term and (4) coincides fully with the classical functional—δW of the potential energy of the linear theory[7-8] ($\xi = v_0 t$). Consequently, in the linear approximation, if $A_2 > 0$, then the configuration is stable relative to a given initial perturbation, but if $A_2 < 0$, it is unstable. As seen from (4), the third term $\sim \lambda^2 t^4$ is always positive, and becomes decisive (in those cases when A_2 is small ("resonances"). Thus, we arrive at the conclusion that even a system that is stable in the linear approximation ($A_2 > 0$) loses stability at an arbitrarily small perturbation ($\lambda \to 0$) if

$$t > t_0 = \sqrt{A_2/2A_4}. \qquad (5)$$

An analysis of the expression for t_0 shows that in the case of equilibrium configuration (3) with a quasihomogeneous current ($N \gg 1$), at an initial perturbation (1) where $m=2$ in the vicinity of the resonance $\mu_0 = \alpha$ we have $t_0 \approx |\alpha - \mu_0|/2\mu_0^2$. Of course, at sufficiently long times, an appreciable role will be played by the discarded terms of the expansion in t. However, as already noted above, the numerical calculation points unambiguously to development of instability, which destroys the initial structure of the pinch (Figs. 1a and 1b). A clear illustration of the foregoing analysis is the time variation of W_k, shown in Fig. 3, and computer-calculated by integrating the system (2). It is seen that in accordance with (4) the value of W_k first decreases slowly, inasmuch as $A_2 > 0$, and then begins to increase rapidly.

It is of interest to note that the considered nonlinear instability develops mainly on account of the thermal energy of the plasma. Indeed, at $\operatorname{div} \mathbf{v}_0 = 0$

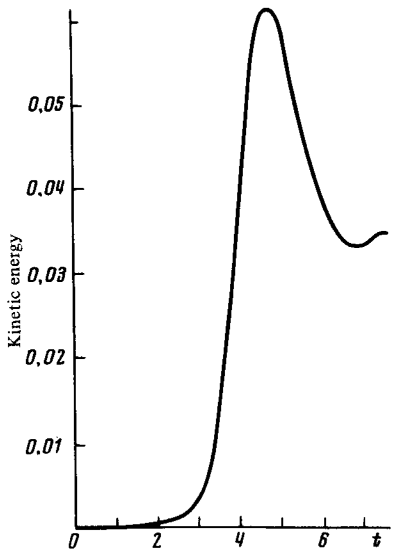

FIG. 3. Kinetic energy W_k for $N=1$; $\mu_0/\alpha=1.4$; $\lambda=0.01$.

(which is the condition for minimization of δW), the expression for W_T, calculated in analogy with W_k, takes the form

$$W_T = \int \frac{p_0 \, d\tau}{\gamma - 1} - \frac{t^4}{8} \int \{ \rho_0 v_2^2 - v_2([j_1 \times B_0] + [j \times_0 B_1])\} \, d\tau. \tag{6}$$

Since the total energy is conserved $W_k + W_T + W_B = \text{const}$, then it follows that in accordance with (4) and (6), in the classical linear approximation, only the magnetic kinetic energy W_b is expended in the development of the instability. However, in the vicinity of $\min\delta W$, if the main contribution to the development of the instability is given by the terms $\sim t^4$ and B_1 can be neglected, then we obtain a qualitatively different result. Namely, in this case

$$\delta W_T = -\frac{t^4}{8} \int \rho_0 v_2^2 \, d\tau = -\int p_1 \operatorname{div} v_2 \, d\tau, \tag{7}$$

i.e., 3/4 of the contribution to δW_k is made by the thermal energy and 1/4 by the magnetic energy.

Thus, we have demonstrated the existence of principally nonlinear instabilities of straight plasma pinches. These instabilities develop at arbitrarily small initial perturbations. Direct application of the results to the toroidal case is not valid, since no account is taken of the number of important factors inherent in toroidal configurations ($\min B$, centrifugal effects, neoclassical diffusion, etc.).

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