

Vortex statistics in rotation helium

V. K. Tkachenko

Institute of Solid State Physics, USSR Academy of Sciences

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The statistical mechanics of a two-dimensional system of vortices (negative temperatures) explain the reverse turning, previously observed by the author, of a float in rotating helium. The problem of the statistical distribution of vortices of like sign in a circular cylinder is solved.

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The principles of statistical mechanics of a two-dimensional system of vortices in a bounded vessel were established by Onsager^[1] in 1949. He has shown that negative temperatures have a meaning for such systems. Indeed, the vortex system is a Hamiltonian system, and its phase volume is finite. When the vortex interaction energy $E \rightarrow \pm \infty$, the phase volume tends to zero. This means that at a certain E_0 the phase volume is maximal, meaning in turn that at $E > E_0$ the system is in a state with negative temperature.

In 1970, vortex systems have attracted the interest of plasma-theory specialists,^[2] inasmuch as a two-dimensional plasma in a strong magnetic field is described by the same equations. This yielded soon a non-trivial result, namely: a system of vortices of positive and negative circulation tends to break up at negative temperature into two clouds, one of vortices with positive circulation and the other with negative circulation.^[3,4] This result follows qualitatively from the fact that at negative temperature the system tends to occupy

the state with the maximum energy, and such a state comprises indeed the two clouds described above. In^[5] it was reported that a float in rotating helium turns in the opposite direction. This result can be explained precisely on the basis of the foregoing considerations. The helium dynamics can evolve, for example at the following manner. At the start of the rotation, a Taylor instability arises in the liquid, i. e., macroscopic vortices, consisting of many elementary ones, are produced along the wall of the rotating cylinder. The planes of these vortices are perpendicular to the rotation axis, and the signs alternate. The interaction energy of the elementary vortices is large, meaning that the temperature is negative. Under the influence of the normal component, the vortices gradually become aligned along the rotation axis, retaining the negative temperature. Condensation of the negative-circulation vortices into a cloud takes place at the center of the vessel, and the float begins to rotate in the opposite direction. The irregularity in the reproducibility of the effect indicates that deviations from the described picture are possible.

The float rotation observed in^[5] was quasistationary, with a velocity several times smaller than the angular velocity of the vessel itself. This uneven rotation of the liquid can also be described statistically.

We consider the following problem: A circular cylinder of radius R contains N positive-circulation vortices. It is required to find their statistical equilibrium distribution. We assume that this distribution is axially symmetrical. We denote the vortex density by $n(r)$, where r is the distance from the axis. Then for the entropy, apart from a constant, we have $S = -2\pi \int_0^R n \ln nr dr$. The system of vortices in the cylinder admits of two first integrals—the energy E and the angular momentum M . If we describe the motion of the liquid macroscopically, then $E = \pi \rho \int_0^R v^2 r dr$, and $M = \pi \Gamma \rho \int_0^R (R^2 - r^2) nr dr$. Here $v(r) = (\Gamma/r) \int_0^r nr dr$ is the velocity, Γ is the circulation of the velocity around the vortex, and ρ is the density of the liquid. We need to find a vortex density distribution $n(r)$ such that the entropy S is maximal for given N , E , and M . To this end it suffices to find the extremum of the quantity $S + (\mu/T)N - E/T + \Omega M/T$, where μ/T , $1/T$, and Ω/T are Lagrange multipliers. Varying $n(r)$ by δn in the interval $(n, r+dr)$, we arrive at an integral equation in $n(r)$, differentiation of which with respect to r yields the simple integro-differential equation

$$n' = \frac{\Gamma^2 \rho n}{T r} \int_0^r nr dr - \Gamma \rho r n \frac{\Omega}{T}.$$

At $T=0$ this equation can have a unique smooth solution $n(r) = 2\Omega/\Gamma$. It is valid for both positive and negative temperatures if the relation $\Omega = NT/2\pi R^2$ is satisfied. But if Ω differs from this value, then the vortex density becomes already uneven.

We consider the case $\Omega=0$. It arises if the angular momentum A is not conserved for some reason (for example, because of roughness of the vessel walls). In this case at $T=\infty$ the distribution of the vortex density is uniform, as before, but as $T \rightarrow +0$ the vortices become concentrated along the vessel walls and the rotation at the center is slowed down. It is possible to describe qualitatively in this manner the rotation of the float, ascribing a certain positive temperature to the vortex system.

- ¹L. Onsager, Nuovo Cimento Suppl. **6**, 279 (1949).
- ²J. B. Taylor and B. McNamara, Phys. Fluids **14**, 1492 (1971).
- ³G. Joyce and D. Montgomery, J. Plasma Physics **10**, 107 (1973).
- ⁴S. F. Edwards and J. B. Taylor, Proc. R. Soc. Lond. **A336**, 257 (1974).
- ⁵V. K. Tkachenko, Zh. Eksp. Teor. Fiz. **67**, 1984 (1974) [Sov. Phys.-JETP **40**, 985 (1974)].