Macroscopic similarity theory in the percolation problem

S. L. Ginzburg

B. P. Konstantinov Institute of Nuclear Physics, USSR Academy of Sciences
(Submitted February 16, 1976)
Pis'ma Zh. Eksp. Teor. Fiz. **23**, No. 6, 342–344 (20 March 1976)

The critical exponents of percolation theory and the correlation functions are calculated and an analog of the equation of state is constructed by using field-theoretical methods.

PACS numbers: 75.30.Jy, 64.60.–i

It was shown with mathematical rigor in[1] that the problem of bonds in percolation theory is equivalent to a second-order phase transition of the S model as $S\to 1$. In[2–4], a similarity hypothesis was formulated with respect to the percolation problem. The authors of[5] attempted to construct a microscopic theory of similarity by ϵ-continuation from six-dimensional space. The S-model Hamiltonian was replaced by another model Hamiltonian. This replacement is not justified, and this seems to be the reason why $\eta<0$ was obtained in[5] (η is the Fisher parameter).

In this article we construct a microscopic similarity theory for the percolation problem by using the renormalization-group method, which is valid also directly for three-dimensional space.[6,7] The critical exponents and the correlation functions are calculated, and the analog of the equation of state is obtained for the order parameter (which in the percolation problem is the power of the infinite cluster[2]). For spaces with dimensions $3 < d < 6$. The Hamiltonian of the S model is of the form[5]

$$H = -\sum_{i,k} J_{ik} P_{\sigma_i \sigma_k} - \omega_0 \sum_i P_{\sigma_i 1} \tag{1}$$

$$P_{\sigma_i \sigma_k} = S\delta_{\sigma_i \sigma_k} - 1$$

J_{ik} is the exchange integral, $\omega_0 = \mu H$, S is the number of components of the S model. The percolation problem is connected with the limit $S \to 1$ with $q = \exp(-2J/T)$ and $x = 2\omega_0/T$, where q is the probability of the bond being broken, x is a formal magnetic field,[1,2] T is the temperature in the partition function of the S model. It can be shown that the correlation function $g(r)$ of the percolation problem[3,4] and the power $P(q)$ of the infinite cluster are equal, in terms of the S model, to

$$g(\mathbf{r}_i - \mathbf{r}_k) = \lim_{S \to 1} \frac{1}{S-1} \langle P_{1\sigma_i} P_{1\sigma_k} \rangle$$

$$P(q) = \lim_{S \to 1} \frac{1}{S-1} \langle P_{1\sigma_i} \rangle . \tag{2}$$

If, starting from (1) and (2), we construct a diagram technique, then it turns out that in the corresponding theory there is triple vertex Γ_3. We, however, will not take it into account. In the scaling region $\Gamma_3(\mathrm{p}=0) \sim \kappa^{6-d/2}$ (p is the momentum, $\kappa = r_c^{-1}$, r_c is the correlation radius). By a method analogous to that developed in[6,7], we obtain the following equation of the renormalization group $3 < d < 6$ for the dimensionless coupling constant g, the function f_R (their definition is given in (3)), and the susceptibility exponent γ:

$$\Gamma_3(\mathrm{p}=0) = a g \kappa^{6-d/2}, \quad f_R = \frac{\partial \kappa^2}{\partial \tau},$$

$$\frac{\partial g}{\partial t} = \Psi(g) = -\frac{6-d}{4} g + (3-S) g^3,$$

$$\frac{\partial \ln f_R}{\partial t} = \xi(g) = (2-S) g^2, \tag{3}$$

$$\gamma = [1 - \xi(g_0)]^{-1}\Big|_{S=1} = \frac{8}{2+d},$$

$$\tau = q - q_c, \quad t = \ln \kappa^2,$$

where q_c is the critical concentration, g_0 is the zero of $\Psi(g)$, and α is a constant. In analogous fashion, at $d=3$ we obtain $\eta \approx 1/12$. The remaining exponents are calculated in accordance with the scaling laws. If we assume $\eta = 0$, then $\nu = 4/(2+d)$, $\beta = 2(d-2)/(2+d)$, and $\Delta = \beta + \gamma = 2$. Computer calculations yield,[2,4] at $d=3$, $\beta = 0.35 \pm 0.05$, $\gamma = 1.69 \pm 0.05$, $\nu = 0.9 \pm 0.05$, and $\Delta = 2.2$

± 0.3, which agrees with our results. The correlation function has the form usual for second-order phase transitions.

The equation of state is written for the second and derivatives $\phi_c(x,\tau)$ and $\chi(x,\tau)$ of the analog of the free energy,[3] as functions of x and τ. Obviously, $\phi_c(0,\tau)=P(q)$ and $\chi(0,\tau)=g(\mathrm{p}=0)$. We write down the equation of state in parametric form in analogy with[8]:

$$h = m - 2\sqrt{\pi}m^2,$$

$$m = \phi_c \chi^{\beta/\gamma}, \quad h = x\chi^{(\beta+\gamma)/\gamma}. \tag{4}$$

This equation at $d>6$ yields the analog of the Landau theory for percolation with exponents $\gamma=1$, $\beta=1$, $\nu=1/2$, and $\eta=0$. Since in our case $\beta+\gamma=2$ even at $d=3$, we can solve (4) in explicit form. This makes it possible to obtain explicit expressions for the thermodynamic functions of the disordered Ising model at low temperatures near $q\sim q_c$ as functions of q and of the external magnetic field. The analog of the exponent $\alpha=-2/5$ of the specific heat determines the singular part of the energy of the ground state $E\sim\tau^{1-\alpha}$.

[1]P. W. Kasteleyn and C. M. Fortuin, J. Phys. Soc. Jap., Suppl. **26**, 11 (1969).
[2]J. W. Essam and K. M. Gwilym, J. Phys. C **4**, L288 (1971).
[3]A. G. Dunn, J. W. Essam, and J. M. Loveluck, J. Phys. C **8**, 743 (1975).
[4]M. E. Levinshteĭn, B. I. Shklovskiĭ, M. S. Shur, and A. L. Éfros, Zh. Eksp. Teor. Fiz. **69**, 386 (1975) [Sov. Phys.-JETP **42**, 197 (1975)].
[5]A. B. Harris, T. C. Lubensky, W. K. Holcomb, and C. Dasqupta, Phys. Rev. Lett. **35**, 327 (1975).
[6]S. L. Ginsburg, Zh. Eksp. Teor. Fiz. **68**, 273 (1975) [Sov. Phys.-JETP **41**, 133 (1975)].
[7]J.-L. Colot, J. A. C. Loodts, and R. Brout, J. Phys. A **8**, 594 (1975).
[8]A. A. Migdal, Zh. Eksp. Teor. Fiz. **62**, 1559 (1972) [Sov. Phys.-JETP **35**, 816 (1972)].