Integrability of the two-dimensional Thirring model

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It is shown that the inverse-problem method can be used in the Thirring model. A recurrence formula is obtained for the sequence of the integrals of motion.

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Interest in two-dimensional models of field theory is greatly increased after the first nontrivial model—the "sine-Gordon" equation—was investigated in detail by the inverse-problem method.[1] In the present paper we consider the Thirring model[2]

$$(-i\partial_\nu\gamma^\nu + m)\psi = \pm g\gamma^\nu\psi(\bar{\psi}\gamma_\nu\psi) \tag{1}$$

where $\nu=0$ and 1, and

$$\gamma^{\circ} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^{1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^{5} = -i\gamma^{\circ}\gamma^{1}, \quad \bar{\psi} = \psi^{+}\gamma^{\circ} \tag{2}$$

—theory of a spinor field with nonzero mass.

We use a generalization of the inverse-problem method proposed by Zakharov,[3] which consists briefly in the following. We consider a pair of operator beams $X=\sum_{s}^{t}A_i\lambda^i$ and $T=\sum_{n}^{m}B_i\lambda^i$, where A_i and B_i are linear operators and λ is a spectral parameter. The requirement

$$[X, T] = 0 \tag{3}$$

is equivalent to vanishing the coefficients of all powers of λ, i.e., it leads to a system of equations. The solution of this system reduces to a study of the direct and inverse problem for the operator beam.

To represent Eq. (1) in the form (3) we can use the following operator beams

$$X = 2i\partial_x \mp ig(\bar{\psi}\gamma_1\psi)\gamma^5 - i\lambda\epsilon + i\lambda^{-1}a - i\frac{m}{2}(\lambda^2 - \lambda^{-2})\gamma^5, \tag{4}$$

$$T = 2i\partial_t \mp ig(\bar{\psi}\gamma_0\psi)\gamma^5 - i\lambda\epsilon - i\lambda^{-1}a - i\frac{m}{2}(\lambda^2 + \lambda^{-2})\gamma^5, \tag{5}$$

where

$$\epsilon = \sqrt{2mg}\begin{pmatrix} 0 & \psi_2^* \\ \pm\psi_2 & 0 \end{pmatrix}, \quad a = \sqrt{2mg}\begin{pmatrix} 0 & \psi_1^* \\ \pm\psi_1 & 0 \end{pmatrix}. \tag{6}$$

It is easy to verify that the Lorentz transformation of these beams is in accord with a linear representation (in this case, like the components of a covariant vector), if together with the vectors $A^0 = A^{0\prime}\cosh b + A^{1\prime}\sinh b$ and $A^1 = A^{1\prime}\cosh b + A^{0\prime}\sinh b$ and the spinors $\psi = [\exp(b\sigma_3/2)\psi']$ we transform also the spectral parameter $\lambda = [\exp(-b/2)]\lambda'$. It can be shown that under gauge transformations the beams transform in accordance with a linear representation of the corresponding group $X = EX'U^{-1}$, $T = UT'U^{-1}$. This is a sufficient condition for the invariance of the system (3) to the Lorentz group and to the group of gauge transformations. We note that the indicated condition makes it possible to narrow down considerably the class of permissible operator beams when searching for physically interesting systems (3) with specified group properties.

All the nonlinear equations solved by the usual inverse-problem method have an important property, viz., the existence of an infinite set of integrals of motion. An analogous property can be established also for the given scheme. The fact that an infinite set of integrals exists follows from the fact that the diagonal elements of the transition matrix, determined in analogy with[4], does not depend on the time. Their expansion in positive powers of λ in the vicinity of zero and in negative powers of λ in the vicinity of infinity generates a denumerable sequence of integrals of motion I_n and I_{-n}. For these we can write down a recurrence formula. For these we can write down a recurrence formula. We define to this end an auxiliary sequence A_n ($n = -1, 0, 1, 2\cdots$) in the following manner:

$$A_{-1} = A_0 = 0$$

$$i\sqrt{\frac{m}{2g}} A_{k+2} + \psi_1 \sum_{i+j=k+1} A_i A_j \mp \psi_1^* \delta_{k,-1} \pm \sqrt{\frac{2g}{m}} \bar{\psi}\gamma^1\psi A_k - \sqrt{\frac{2}{mg}} A_{kx} \tag{7}$$

$$\pm \psi_2^* \delta_{k,1} - \psi_2 \sum_{i+j=k-1} A_i A_j - i\sqrt{\frac{m}{2g}} A_{k-2} = 0,$$

then integrals of motion I_n take the form

$$I_n = i\sqrt{\frac{mg}{2}} \int_{-\infty}^{\infty} (\psi_2 A_{n-1} - \psi_1 A_{n+1})\, dx. \tag{8}$$

The integrals I_{-n} corresponding to the expansion in inverse powers of λ, are connected with I_n by the relation

$$I_{-n} = \int_{-\infty}^{\infty} F(\bar{\psi}\gamma^0, \gamma^0\psi, -x)\, dx, \tag{9}$$

where F is the density of the integral $I_n = \int F(\bar{\psi}, \psi, x)\, dx$. From the fact that the transition matrix is unimodular it follows that all the integrals of motion are real. We present some of the first terms of the sequence

$$I_0 = I_{-0} = \mp \frac{g}{2} \int_{-\infty}^{\infty} \bar{\psi}\gamma^0\psi\, dx, \tag{10a}$$

$$I_2 = \pm \frac{2g}{m} \int_{-\infty}^{\infty} \left[-\frac{i}{2} \psi_1^* \overleftrightarrow{\partial}_x \psi_1 + \frac{m}{2} \bar{\psi}\psi \mp \frac{g}{4} (\bar{\psi}\gamma_\nu\psi)(\bar{\psi}\gamma^\nu\psi) \right] dx, \tag{10b}$$

$$I_{-2} = \pm \frac{2g}{m} \int_{-\infty}^{\infty} \left[+ \frac{i}{2} \psi_2^* \overleftrightarrow{\partial}_x \psi_2 + \frac{m}{2} \bar{\psi}\psi \mp \frac{g}{4} (\bar{\psi}\gamma_\nu\psi)(\bar{\psi}\gamma^\nu\psi) \right] dx, \tag{10c}$$

$$I_4 = \mp \frac{g}{m^2} \int [\, m^2 \bar{\psi}\gamma^0\psi - im\bar{\psi} \overleftrightarrow{\partial}_x \psi + 4|\psi_{1x}|^2 \mp 2mg (\bar{\psi}\gamma^0\psi)(\bar{\psi}\psi)$$

$$\mp 4g (\bar{\psi}\gamma^0\psi)\psi_1^* \partial_x \psi_1 + g^2 (\bar{\psi}\gamma^0\psi)(\bar{\psi}\gamma_\nu\psi)(\bar{\psi}\gamma^\nu\psi)\,]\, dx. \tag{10d}$$

All the odd integrals are equal to zero.

The charge conservation law corresponds to the integral I_0. The energy-momentum vector is expressed in terms of I_2

$$P_0 = \pm \frac{m}{2g} (I_2 + I_{-2}), \tag{11a}$$

$$P_1 = \pm \frac{m}{2g} (I_2 - I_{-2}). \tag{11b}$$

We note in conclusion that several of the integrals of motion for Eq. (1) were obtained by Aref'eva. [5]

A detailed investigation of Eq. (1) is quite cumbersome and will be published elsewhere.

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