A new approach in the problem of three and more bodies

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A method is proposed for solving the problem of a small number (more than two) of bodies, based on the law of evolution of the system as the coupling constants are varied. In the problem of elastic (n, d) scattering, even the lowest approximation of the method yields unitary expressions that agree with experiment.

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Among the singularities of the few-body problem (FBP), which make its solution difficult, it is customary to single out the ambiguity of the Lippmann-Schwinger equation and the non-Fredholm type of its kernel. However these singularities can be easily eliminated by changing over, for example, to the known Faddeev equation.[1] More important from the practical point of view is another difficulty of the FBP, namely the divergence of the Born series. In the problem considered below, that of neutron scattering by deuterons, the first Born terms deviate from experiment by large factors and disagree greatly with the unitarity condition.[1)] This makes it necessary either to carry out a laborious numerical integration of the Fadeev equations, or to use an artificial "unitarization" procedure, or in general to dispense with the dynamic description and resort to the general requirements of unitarity and analyticity.[2]

The proposed new approach to the FBP, which is free of this difficulty, was used earlier in quantum field theory in the two-body problem.[3] It is based on the law of evolution of the system not as the time is varied, as is customary, but the coupling constant g, defined as the proportionality factor in the interaction Hamiltonian

$$H_{int} = gV. \tag{1}$$

The problem reduces to a differential equations in g which is of relatively simple form, and which organically include bound states. The main feature of interest to us in the proposed method lies in the exact satisfaction of the unitarity condition (and causality) at each stage of the successive approximations.

As applied to the problem of elastic (n,d) scattering in the absence of a three-particle bound state (quartet state and higher partial waves of the doublet scattering) even the lowest approximation of the method agrees well with experiment.

1. The equations of the proposed method are of the form[3]

$$\frac{df_{mn}}{dg} = -\frac{m}{2\pi} V_{mn} - 2\pi i \sum_s f_{ms} V_{sn} \delta(E_s - E_n) \tag{2}$$

$$\frac{dV_{mn}}{dg} = \sum_s V_{ms} V_{sn} \left(\frac{1}{E_m - E_s - i\delta} + \frac{1}{E_n - E_s + i\delta} \right), \tag{3}$$

where f_{mn} is the scattering amplitude for the transition $n \to m$ $(E_m = E_n)$, V_{mn} is the matrix element of the Heisenberg operator V (see (1)), m is the reduced mass, and satisfaction of the momentum conservation law is implied. The sum includes a complete set of states, including the states of the discrete spectrum, for which

$$dE_n / dg = V_{nn}. \tag{4}$$

For elastic two-particle reactions, Eq. (2) can be explicitly solved by partial-wave expansion, and this yields the usual parametrization of the amplitude $f_l(t) = [\exp(2i\delta_l(k)) - 1]/(2ik)$, where

$$\frac{d\delta_l(k)}{dg} = -\frac{mk}{2\pi} V_l(k), \qquad V_l(k) \equiv \langle \mathbf{k}_1 - \mathbf{k} | V | \mathbf{k}'_1 - \mathbf{k}' \rangle_l \,|_{k=k'}. \tag{5}$$

Although Eq. (3) cannot be solved in general form, each term of its iteration series makes a unitary (by virtue of the hermiticity of V_{mn}) and causal (by virtue of the circuiting rule in (3)) contribution to the scattering amplitude. According to (5), the matter reduces to an iteration series for the scattering phase shift, which converges rapidly in the absence of many-particle bound states.

2. The lowest approximation of the method corresponds to the diagram of Fig. 1a for the right-hand side of (3), and the corresponding expression for the phase shift of the (n,d) scattering can be written in the form

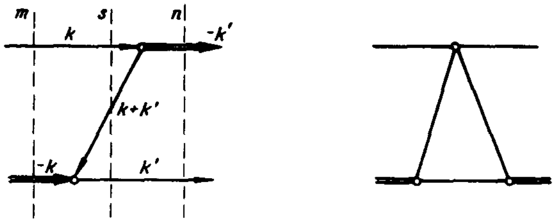

FIG. 1.

$$\delta_l(k) = -\frac{16}{3}\left(\frac{k}{q}(q^2+\kappa^2)\int_\kappa^\infty \frac{d\kappa\kappa}{(q^2+\kappa^2)^2}\bar{\delta}(q)\right)_l . \qquad (6)$$

Here $\bar{\delta}$ is the phase shift of the two-particle s-scattering, $\kappa^2/2m$ is the binding energy of the deuteron, $2 = |\mathbf{k}+\mathbf{k}'/2| = |\mathbf{k}'+\mathbf{k}/2|$ is the momentum transferred during the breakup and formation of the deuteron. The expression for the s-scattering length is

$$a_0 = -\lim_{k\to 0}\delta_0(k)/k = -\frac{16}{3}\kappa^2\int_\kappa^\infty \frac{d\kappa}{\kappa^3}\bar{a} . \qquad (7)$$

where $\bar{a}$ is the two-particle scattering length.

By way of illustration, we indicate that the two-particle interaction is assumed to be separable (see below). This makes it possible to replace the product of the matrix element V for the disintegration and formation of the deuteron by an analogous product for the two-particle scattering and "deuteron—deuteron" transition, and subsequently, according to (4) and (5), by the products of derivatives of the binding energy of the deutron and of the phase shift of the two-particle scattering with respect to g. The resultant integrals with respect to g are replaced by integrals with respect to the quantity κ. The boundary conditions with respect to g are chosen in accordance with the statement made in the footnote.

3. The concrete calculation was carried out using the well-known separable two-particle Yamaguchi potential

$$< \mathbf{k}_1 - \mathbf{k} | V | \mathbf{k}_1' - \mathbf{k}' > = 4\pi\nu^*(k')\nu(k), \quad \nu(k) = \gamma^2/(k^2+\gamma^2),$$

which acts only in the s state (the reciprocal effective radius $\gamma = 1.44\ F^{-1}$ is large in comparison with the reciprocal scattering length $\kappa = 0.23\ F^{-1}$). Introducing the notation $\xi \equiv k/\kappa$ and $\alpha \equiv \kappa/\gamma \ll 1$, we have

$$a = \sqrt{|g|\gamma/2} - 1, \quad \bar{a} = -\frac{2(1+a)^2}{\kappa(2+a)}, \quad \mathrm{tg}\,\bar{\delta}(k) = -2a\xi(1+a)^2 / [(a^2\xi^2+1)^2$$

$$+ (1+a)^2(a^2\xi^2-1)] .$$

From this, in accordance with (7), the length of the quartet s-scattering is

$$^4a_0 = \frac{16}{9\kappa}\left(1 - \frac{3}{4}a + \frac{3}{4}a^2 + \ldots\right) = 6.75\ F;$$

The experimental value is 6.35 F. The expressions for the scattering phase shifts (see (6)) are quite cumbersome and we present only one of them,

$$^4\delta_0(k) = -\frac{8}{3\kappa}\left[\left(\frac{3}{4\xi} + \frac{1}{3\xi^3}\right)\operatorname{arctg}\left(\frac{3\xi}{2}\right) - \left(\frac{1}{4\xi} + \frac{1}{\xi^3}\right)\operatorname{arctg}\left(\frac{\xi}{2}\right)\right] + \ldots .$$

Comparison with experiment is shown in Fig. 2 and indicates sufficiently good agreement.[4]

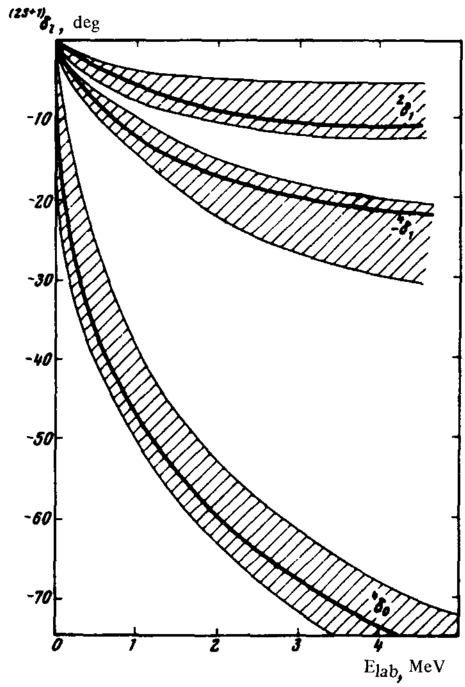

FIG. 2.

The convergence of the method is seen from the estimate of the contribution made to the scattering length by the next higher approximation (see Fig. 1b). This contribution amounts to (128/27 κ) [ln(9/8) − 1/12], and is smaller by one order of magnitude than the figure give above. We emphasize that in the usual Born series the ratio of the contribution of the diagrams of Fig. 1 is reversed.

4. A detailed exposition of the results of this study will be published. The description of the doublet s scattering when there is a bound state (triton) will also be the subject of a separate paper.

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[1)]This is apparently connected with the "looseness" of the deuteron, thanks to which it interacts strongly with the third particle. In the inverse limiting case $g \to \infty$ (vide infra[1]) the deuteron binding energy is proportional to g^2 and is large in comparison with its interaction with the third particle, which is of order g. There is therefore neither virtual disintegration of the deuteron nor any (n, d) interaction, in the limit under consideration.

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