›preading of phonon beam in superfluid helium

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The propagation of a narrow low-intensity phonon beam in superfluid helium at $T < 0.6°K$ is considered. The law governing the spreading of the beam is obtained.

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Mills, Sherlock, and Wyatt,[1] with an aim at determining the phonon dis-ersion law in superfluid helium, have recently performed experiments on the preading of a narrow beam of phonons emitted by a heater into a helium bath having a temperature $\lesssim 0.1\,°K$. The purpose of our present study was to calculate this effect for the case of a low-intensity beam.

It is known that at temperatures below 0.6 °K only the phonons are responsible for all the kinetic phenomena. We assume the pressure not to be too large, so that the phonons have a decay spectrum. Under these conditions we can use the fact that by virtue of the small deviation of the phonon dispersion in the helium from linearity, the relaxation of the phonons propagating along this direction is much more rapid than the relaxation between the directions,[2-4] which we shall call transverse relaxation. Since the beam spreading is connected just with the transverse relaxation, the phonon distribution along any given direction can be regarded as being in equilibrium and can be characterized by a temperature $\Theta(\vec{\kappa})$ that depends on the direction $\vec{\kappa} = \mathbf{k}/k$, where $\mathbf{k}$ is the phonon wave vector. The energy of the phonons propagating along a give direction, $\mathcal{E}(\kappa) = (\pi/120)\Theta^4/\hbar c)^3$ satisfies the equation

$$\frac{\partial \mathcal{E}}{\partial t} + c\kappa_i \frac{\partial \mathcal{E}}{\partial x_i} = \left[\frac{\partial \mathcal{E}}{\partial t}\right]_{\text{coll}}, \quad (1)$$

which can be obtained by multiplying the kinetic equation for the phonon distribution function by $\hbar c k^3\, dk/(2\pi)^3$ and integrating with respect to k from 0 to ∞. The form of the transverse-relaxation operator $[\partial\mathcal{E}/\partial t]_{\text{coll}}$ was determined in[4]

In this paper we assume that the beam intensity is so low that the deviation of the temperature in the beam from its value in the bath is relatively small, i.e., $\Theta = T(1+Z)$, $|Z| \ll 1$. In the approximation linear in Z, under stationary conditions, Eq. (1) takes the form

$$\kappa_i \frac{\partial Z}{\partial x_i} = -\frac{1}{4c\tau_\perp} l^2(l^2+2)Z, \quad (2)$$

where $\tau_\perp$ is the transverse relaxation time, $l_i = e_{ijm}k_j\,\partial/\partial k_m$, and e_{ijm} is a fully antisymmetrical tensor of third rank. We are interested only in that beam-spreading region where the beam remains still narrow enough, i.e.,

$$\kappa_x,\ \kappa_y << \kappa_z \approx 1, \quad (3)$$

where the z axis is directed along the beam propagation direction. Then $l_x \approx -\partial/\partial\kappa_y$, $l_y \approx \partial/\partial\kappa_x$, $l_z \ll l_x, l_y$ and (2) reduces to

$$\frac{\partial Z}{\partial z} + \kappa_\lambda \frac{\partial Z}{\partial x_\lambda} = -\frac{1}{4c\tau_\perp}\left(\frac{\partial^2}{\partial \kappa_\lambda^2}\right)^2 Z, \quad (4)$$

where the subscript λ takes on the values 1 and 2. This equation must be solved with the initial condition $Z = Z_0\delta(\vec{\kappa}_\perp)\delta(\mathbf{x}_\perp)$, where $\vec{\kappa}_\perp$ and $\mathbf{x}_\perp$ are two-dimensional vectors in the planes $(\kappa_x\kappa_y)$ and (xy) respectively. The quantity Z_0 is connected with the total energy flux S_0 in the beam by the relation $S_0 = (\pi/30)Z_0T^4/\hbar^3c^2$. As $x_\lambda \to \pm\infty$ or $k_\lambda \to \pm\infty$, the value of Z should tend to zero.

With the aid of a Fourier transformation in the variables x, y, κ_x, and κ_y, Eq. (4) reduces to a partial differential equation of first order, which can be solved without difficulty. As a result we have

$$Z = \frac{Z_0}{(2\pi)^3}\int_0^\infty \rho\,d\rho \int_0^\infty \sigma\,d\sigma \int_0^{2\pi} d\psi\, I_0\left(\sqrt{x_\perp^2\rho^2 + \kappa_\perp^2\sigma^2 + 2x_\perp\rho\kappa_\perp\sigma\cos(\phi-\psi)}\right)R,$$

where

$$R = \exp\left\{-\frac{z}{4c\tau_\perp}\left[\sigma^4 - 2\sigma^3\rho z\cos\psi + \frac{2}{3}\sigma^2\rho^2 z^2(1+2\cos^2\psi)\right.\right.$$
$$\left.\left. - \sigma\rho^3 z^3\cos\psi + \frac{1}{5}\rho^4 z^4\right]\right\} \quad (6)$$

is the angle between the vectors $\mathbf{x}_\perp$ and $\kappa_\perp$.

It is easy to determine from (5) the dependence of the energy flux density in ıe beam on the distance to its center

$$S(z, x_\perp) = \frac{S_0}{2\pi}\left(\frac{20c\,\tau_\perp}{z^5}\right)^{1/2} f\left[\left(\frac{20c\,\tau_\perp}{z^5}\right)^{1/4} x_\perp\right], \quad (7)$$

here

$$f(x) = \int_0^\infty I_0(xy)e^{-y^4}y\,dy\,. \quad (8)$$

x) has at $x \gg 1$ the asymptotic form

$$f(x) \sim \frac{1}{2\sqrt{3}}\left(\frac{4}{x}\right)^{2/3}\exp\left[-\frac{3}{2}\left(\frac{x}{4}\right)^{4/3}\right]\cos\left[\frac{3\sqrt{3}}{2}\left(\frac{x}{4}\right)^{4/3} - \frac{\pi}{3}\right]. \quad (9)$$

he oscillations that appear here are of purely mathematical origin. They are ›nnected with the high order of the differential operator in the right-hand side ˙ (4) (no such oscillations would occur for a second-order operator).

Expression (5) is in fact the Green's function of Eq. (4). It is easy to in- ›stigate with its aid the spreading of a phonon beam with an initial distribution at is sufficiently narrow in k and in x space, but is of arbitrary shape. It can › noted here that the scales of $x_\perp$ and $\kappa_\perp$ over which the Green's function (5) ıries are of the order of $z(z/c\tau_\perp)^{1/4}$ and $(z/c\tau_\perp)^{1/4}$, respectively. For this ›ason, a phonon beam with initial distribution of arbitrary shape is described ˙ the distribution (5) at sufficiently large values of z_0.

It must be borne in mind, however, that the quantity $\int \kappa_\perp^2 Z d^2\vec{\kappa}_\perp d^2\mathbf{x}_\perp$, which by rtue of (4) is independent of z, is identically equal to zero for the distribution). A finite value is obtained for this quantity in the case of a narrow but not like source if the next term of the asymptotic expansion in reciprocal powers z is taken into account in addition to (5).

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