Hamiltonian formalism in classical and two-fluid hydrodynamics

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A Hamiltonian formalism is developed for the hydrodynamics of a superfluid liquid. This formalism leads to a hitherto-unknown Hamiltonian formalism for non-isentropic flows of a classical liquid.

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The equations of two-fluid hydrodynamics can be written in Hamiltonian form if the Hamiltonian H is chosen to be the energy of the liquid in an immobile coordinate system

$$H = \int\left[\frac{\rho \mathbf{v}_s^2}{2} + \mathbf{p}\mathbf{v}_s + \epsilon(\rho, S, \mathbf{p})\right] dV \,, \tag{1}$$

where ρ is the density, $\mathbf{v}_s$ is the velocity of the superfluid component, $\mathbf{p}$ is the momentum per unit volume of the liquid in a reference frame moving with velocity $\mathbf{v}_s$, and $\epsilon(\rho, S, \mathbf{p})$ is the energy per unit volume of the liquid in the same coordinate frame and is defined by the thermodynamic identity

$$d\epsilon = TdS + \mu d\rho + (\mathbf{v}_n - \mathbf{v}_s)\, d\mathbf{p} \,. \tag{2}$$

The momentum j per unit volume of the liquid in the immobile system is equal to

$$\mathbf{j} = \rho \mathbf{v}_s + \mathbf{p} = \rho_s \mathbf{v}_s + \rho_n \mathbf{v}_n . \tag{3}$$

The Hamiltonian variables are the three canonically conjugate pairs (ρ, α), (S, β), and the Clebsch variables[1] (f, γ).

In addition to the already defined quantities ρ and S, we introduce four new quantities α, β, γ, and f, the physical meaning of which is the following. The quantity α defines the velocity of the superfluid component:

$$\mathbf{v}_s = \nabla \alpha . \tag{4}$$

The quantities β, γ, and f define the relative-motion momentum

$$\mathbf{p} = S \nabla \beta + f \nabla \gamma . \tag{5}$$

Expression (1) acquires the meaning of a Hamiltonian if (4) and (5) are substituted in (1). The Hamilton equations take the usual form:

$$\dot{\rho} = \frac{\delta H}{\delta \alpha} = - \operatorname{div} \mathbf{j} , \tag{6}$$

$$\dot{S} = \frac{\delta H}{\delta \beta} = - \operatorname{div} (S \mathbf{v}_n) , \tag{7}$$

$$\dot{f} = \frac{\delta H}{\delta \gamma} = - \operatorname{div} (f \mathbf{v}_n) , \tag{8}$$

$$\dot{\alpha} = - \frac{\delta H}{\delta \rho} = - \left(\mu + \frac{\mathbf{v}_s^2}{2}\right) \tag{9}$$

$$\dot{\beta} = - \frac{\delta H}{\delta S} = - T - \mathbf{v}_n \nabla \beta , \tag{10}$$

$$\dot{\gamma} = - \frac{\delta H}{\delta f} = - \mathbf{v}_n \nabla \gamma . \tag{11}$$

Equations (6) and (7) are the known continuity equations for the density and the entropy. Equation (9) is the equation of the superfluid motion, as can be easily verified by taking the gradients of both sides of the equation. Differentiating (5) with respect to time and using (6), (7), (10), and (11) we obtain the known equation of the relative motion[2]:

$$\dot{\mathbf{p}} + \mathbf{p} \operatorname{div} \mathbf{v}_n + \nabla (\mathbf{p} \mathbf{v}_n) - [\mathbf{v}_n \times \operatorname{curl} \mathbf{p}] + S \nabla T = 0 . \tag{12}$$

Combining (9) and (12), we obtain

$$\dot{j}_n + \frac{\partial \Pi_{ik}}{\partial x_k} = 0 , \tag{13}$$

$$\Pi_{ik} = \rho v_{si} v_{sk} + v_{si} p_k + v_{nk} p_i + p \delta_{ik} , \tag{14}$$

where the pressure p is defined by

$$p = - \epsilon + TS + \mu \rho + (\mathbf{v}_n - \mathbf{v}_s) \mathbf{p} . \tag{15}$$

We have thus verified the correctness of the choice of the canonically conjugate variables. The variables β, f, and γ are needed to describe the three independent components of the vector p. We note that the same variables can be used to describe the motion of a classical liquid. The momentum per unit mass is then

$$\mathbf{j} = \rho \mathbf{v} = \rho \nabla \alpha + S \nabla \beta + f \nabla \gamma , \tag{16}$$

and the Hamiltonian must be chosen in the form

$$H = \int \left[\frac{\rho v^2}{2} + \epsilon(\rho, S) \right] dV . \tag{17}$$

The Hamilton equations for a classical liquid are of the same form as (6)—(11), except that we must put in them $\mathbf{v}_n = \mathbf{v}_s = \mathbf{v}$. Equations (6) and (7) then go over into the usual continuity and entropy-conservation equations, while (13) becomes Euler's equation. The usual Clebsch formulation[1] makes it possible to obtain the hydrodynamic equations in Hamiltonian form only for the case of barotropic flow, when ϵ depends only on ρ. Our representation of the momentum (13) contains an extra term $S\nabla\beta$ in comparison with the usual Clebsch expression, so that nonbarotropic flows can also be described. To describe the vector $\mathbf{j}$ in classical hydrodynamics it suffices to have three independent quantities, so that one of the four quantities introduced by us, say f, is superfluous. In two-velocity hydrodynamics this function is independent of the others Inasmuch as in classical hydrodynamics there are five independent quantities (ρ, S, and the vector $\mathbf{j}$), it follows from the six Hamilton equations that one of the quantities, e.g., f, is not independent.

The Hamiltonian (1) can be obtained in the usual manner from the Lagrangian formulation of the two-fluid hydrodynamics, which was proposed by one of us in[3].

The Hamiltonian formalism in two-fluid hydrodynamics makes it easy to prove the following theorem. Assume that at some initial instant $t = t_0$ the quantity curl $\mathbf{p}/S$ is equal to zero in all space. Then it remains equal to zero in all the succeeding instants of time. Indeed, it follows from (8) that the quantity f possesses this property. From this and from Eq. (5) we obtain the statement made above. Thus, in two-fluid hydrodynamics there exists an analog of the circulation theorem known in the hydrodynamics of an ideal liquid.

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